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A Thesis presented for the degree of
Doctor of Philosophy

Centre for Particle Physics Theory
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Dedicated

To my wife, The best of my life

To my children, sweet fruits of heaven

To my parents

and of course

To whom it may concern !

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Abstract

In this thesis, we firstly extend elements and periodicity properties of the theta function theory to functions that represent a wider domain of symmetries and properties, graded with different amounts of $p \geq 1$, $p \in \mathbb{N}$. Unlike theta functions, these generalised, "*higher-level Appell functions*" $\mathcal{K}_p$ satisfy *open quasiperiodicity* relations, with additive theta function terms emerging as *violating* terms of open quasiperiodic $\mathcal{K}_p$'s. We evaluate the S and T modular transformations of these functions and show that the S-transform of $\mathcal{K}_p$ does not just give back $\mathcal{K}_p$, but also includes p additional ϑ-functions which are precisely those violating the quasiperiodicity of Appell functions. This sets a new pattern of modular group representations on functions that are not double quasiperiodic. While calculating the S-transform of $\mathcal{K}_p$, a newly arising function, namely $\Phi(\tau, \mu)$ will be also thoroughly analysed.

As two interesting applications, we firstly study the modular group action on unitary and on an admissible class of non-unitary $N = 2$ characters which are not periodic under the spectral flow and cannot therefore be rationally expressed through theta functions. Secondly we continue this study for the admissible representation of the affine Lie superalgebra $\widehat{s\ell}(2|1)$. We see in the final result for both cases that the functions $\Lambda(\tau, \nu)$ are the "*violating*" terms of unitary calculations. We lastly confirm all our results by some sets of consistency checks including an essential residue calculation. We believe this new way of using Appell functions, could be used for any other algebraic structure whose characters can be rewritten in terms of higher-level Appell functions.

Declaration

The work in this thesis is based on research carried out at the Centre for particle physics theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. However it must be still said that much new material all through the current thesis have been achieved by very fruitful discussions and collaborations with my supervisor and other people who have been listed in my acknowledgement page.

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Chapter 1

Introduction

Problem-solving in science, and in fundamental physics in particular, necessitates the use and development of appropriate mathematical tools. Once the problem is solved, the scientist stands back and often tries to revisit the subject in a broader perspective. By studying the context in which the problem was initially tackled, he or she is led to seek generalisations of the mathematical tools themselves. Solving the general case may sometimes appear at first to be irrelevant to the description of the world as we see it, but it always provides valuable information on the special cases it reduces to and which are more immediately relevant. The less straightforward a generalisation is, the more it tells on 'miraculous' special cases.

The original motivation behind this thesis was to investigate whether one could build a consistent conformal field theory whose symmetry is the superconformal $N = 2$ algebra at central charge

$$c = 3(1 - \frac{2p}{u}), \qquad u = 3, 4, ..., \; p = 2, 3, ..., u \text{ and } p \text{ coprime.} \tag{1.1}$$

The answer to this question is still under debate, but we achieve an important step toward that ultimate goal: we calculate the behaviour of the corresponding irreducible characters under the modular group and highlight the mathematical structure behind that behaviour. In fact, our thesis goes much beyond the particulars of $N = 2$ superconformal algebras. The tools developed, namely the *higher-level Appell* *functions* [1], and the techniques employed are of use in a wide range of situations where the characters are not quasiperiodic in variables belonging to $\mathbb{C}$.

It should be clear that our work has much more to do with the analysis of complex functions (which happen to describe characters of particular representations of infinite-dimensional algebras) than with aspects of the potentially related conformal field theories. We therefore only touch upon conformal field theory considerations here, and put the emphasis on the highly non-trivial mathematics involved.

Two-dimensional conformal symmetry and its supersymmetric generalisations have been extremely popular over the last twenty years as they underlie the description of String Theory in the context of Elementary Particle Theory, but also the description of critical phenomena in Statistical Mechanics [2]. These theories are particularly successful in describing Nature in regimes where they are unitary and minimal. The constraint of *unitarity* for a conformal field theory is that of the absence of negative norm states in the theory. The physical implication is that two-point correlation functions of primary fields (except for the identity operator) fall off with distance. Such behaviour however should not be expected in *all* physical systems described by a two-dimensional model. For instance, the spin system with short-range interactions known as the Yang-Lee edge singularity, and also polymers, have phases described by non-unitary models. So the unitarity condition should not be confused with a physical condition. *Minimality* constrains the conformal field theory to have a finite number of local fields with well-defined scaling behaviour. In the case of non-supersymmetric conformal field theory for instance, where the underlying symmetry is the Virasoro algebra, the minimal sectors are described by central charges of the form

$$c = 1 - 6\frac{(p-p')^2}{pp'}, \qquad p \text{ and } p' \text{ coprime}, \tag{1.2}$$

[1]These functions at level one or higher should not be confused with the hypergeometric functions F_1, F_2, F_3 and F_4 familiar to those calculating loop corrections to Feynman diagrams, and which bear the same name.

[2]The subject is well-documented and the literature vast. A good starting point could be [15]

and for each choice of pair (p, p'), there is a finite number of primary fields with conformal dimension

$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'} \tag{1.3}$$

with r, s in the ranges

$$1 \leq r < p', 1 \leq s < p,\ pr < p's. \tag{1.4}$$

Unitarity requires a further constraint on the pair of coprime integers p and p': one must ensure that $p' = p + 1$. Primary fields $\phi_{r,s}(z)$ with $z \in \mathbb{C}$ are associated with irreducible highest weight characters. The latter are complex functions of one or more variables (depending on the infinite-dimensional algebra considered) counting the number of independent positive norm states obtained by application of the (negative) modes of generators on the highest weight state which is in one-to-one correspondence with a given primary field.

In the context of String Theory as well as critical phenomena in Statistical Mechanics, one is led to consider conformal field theories defined on a torus rather than on the whole complex plane. In critical phenomena, the torus allows the imposition of periodic boundary conditions in two directions of the complex plane. In interactive String Theory, the torus is the first of a series of Riemann surfaces of non-zero genus which describe the worldsheet of closed strings splitting and joining together again. It is important for the consistency of such theories that the correlation functions are indifferent to the parametrisations of these Riemann surfaces. Surfaces of genus at least one have a set of complex parameters or moduli which can be moved by transformations not continuously connected to the identity. Such transformations change the values of moduli but not the shape of the surface, and they are called *modular transformations*. The torus has one modulus, $\tau \in \mathbb{C}$, and modular invariance of the vacuum to vacuum amplitude on a torus (the partition function) is a very strong constraint on a conformal field theory. The set of modular transformations of a torus are the Möbius transformations $\mathcal{M}$ on τ,

$$\mathcal{M} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \mathbb{C} \to \mathbb{C} : \tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d, \in \mathbb{Z},\ ad - bc = 1. \tag{1.5}$$

The 2×2 matrices in (1.5) form the group $SL(2, \mathbb{Z})$. More precisely, since $-a, -b, -c, -d$ lead to the same Möbius transformation, we actually restrict the modular group to $PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z})/Z_2$. All Möbius transformations are generated by the two following transformations:

$$\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

with the relations,

$$\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = (\mathcal{T}\mathcal{S})^3 = \mathcal{C}, \tag{1.6}$$

where $\mathcal{C} = -I$ (and obviously, $\mathcal{C}^2 = I$). In other words, the two generators are,

$$\begin{aligned} S: \ \tau \ &\rightarrow \ -\frac{1}{\tau}, \\ T: \ \tau \ &\rightarrow \ \tau + 1 . \end{aligned} \tag{1.7}$$

Torus partition functions may be expressed as bilinear combinations of characters of irreducible representations of the symmetry algebra considered. The knowledge of such characters and of how they transform under the modular group is therefore crucial as a first step to build the modular invariant partition functions.

We illustrate the above statement in the case of a well-known simple example: the unitary minimal Virasoro model at central charge $c = \frac{1}{2}$ (Ising model). The irreducible characters of minimal Virasoro theories at central charge (1.2) and conformal dimension (1.3) are conventionally labelled $\chi_{r,s}(\tau)$[3]. The expression for the partition function of such models reads,

$$Z(\tau) = \sum_{(r,s),(t,u)\in E_{p,p'}} n_{rs;tu}\chi_{r,s}(\tau)\bar{\chi}_{t,u}(\bar{\tau}), \tag{1.8}$$

where $E_{p,p'}$ denotes the set of pairs (r, s) in the range (1.4). The multiplicities $n_{rs;tu}$ of occurrence of the corresponding left-right representation modules are non-negative integers and the identity is non-degenerate (i.e. $n_{1,1;1,1} = 1$) in a *physical* partition function. A sufficient condition for the above partition function to be modular invariant is that it satisfies,

$$Z(\tau + 1) = Z(\tau), \qquad Z(-\frac{1}{\tau}) = Z(\tau), \tag{1.9}$$

[3]For an explicit expression, see for instance [15].

i.e. that it be invariant under the transformations T and S (1.7). It has been long known that these transformations act linearly on the basis of minimal Virasoro characters, namely

$$\begin{aligned} \chi_{r,s}(\tau+1) &= \sum_{(\rho,\sigma)\in E_{p,p'}} \mathcal{T}_{rs,\rho\sigma}\chi_{\rho,\sigma}(\tau) \\ \chi_{r,s}(-\frac{1}{\tau}) &= \sum_{(\rho,\sigma)\in E_{p,p'}} \mathcal{S}_{rs,\rho\sigma}\chi_{\rho,\sigma}(\tau), \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} \mathcal{T}_{rs,\rho\sigma} &= \delta_{r,\rho}\,\delta_{s,\sigma}e^{2i\pi(h_{r,s}-\frac{c}{24})} \\ \mathcal{S}_{rs,\rho\sigma} &= 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma}\sin(\pi\frac{p}{p'}r\rho)\sin(\pi\frac{p}{p'}s\sigma) \end{aligned} \tag{1.11}$$

are unitary matrices. Constructing a modular-invariant partition function amounts to finding a set of multiplicities $n_{rs;tu}$ such that

$$\begin{aligned} n_{1,1;1,1} &= 1 \\ n\,\mathcal{T} &= \mathcal{T}\,n \\ n\,\mathcal{S} &= \mathcal{S}\,n, \end{aligned} \tag{1.12}$$

where the last two conditions express (in matrix form) the invariance of the partition function under T and S. For the Ising model ($p=3, p'=4$), the three primary fields are $\phi_{1,1}(z), \phi_{1,2}(z)$ and $\phi_{2,2}(z)$ (identity, energy and spin fields respectively) and the matrices $\mathcal{T}$ and $\mathcal{S}$ are given by,

$$\mathcal{T} = \begin{pmatrix} e^{-\frac{i\pi}{24}} & 0 & 0 \\ 0 & e^{\frac{23i\pi}{24}} & 0 \\ 0 & 0 & e^{\frac{i\pi}{12}} \end{pmatrix} \tag{1.13}$$

and

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \tag{1.14}$$

The simplest (and in this case, the unique) modular invariant partition function is easily constructed as the following diagonal invariant,

$$Z^{Ising} = |\chi_{1,1}|^2 + |\chi_{1,2}|^2 + |\chi_{2,2}|^2. \tag{1.15}$$

In fact, such diagonal invariants exist for all minimal Virasoro models, thanks to the unitarity of $\mathcal{S}$. But in general the weakest condition of T invariance allows for a bit more freedom than $h = \bar{h}$, namely T invariance is preserved if $h \equiv \bar{h} \bmod 1$. It is then possible to construct non-diagonal modular invariant partition functions containing a subset of primary fields $\phi_{r,s}$, $(r,s) \in E_{p,p'}$ [4], using (1.10) and (1.11).

Strictly speaking, the terminology 'minimal non unitary' in the context of $N = 2$ superconformal algebra is slightly misleading, and we choose to refer to such models as 'admissible' instead. Indeed, the sectors of interest to us are those with central charge c given in (1.1), and for each pair of coprime integers (p, u), the possible primary fields have conformal dimension

$$h_{r,s,\theta} = s - 1 - (r-1)\frac{p}{u} + \theta(1 - \frac{2p}{u}), \tag{1.16}$$

with r, s, θ in the ranges

$$1 \leq r \leq u-1, \qquad 1-p \leq s \leq p, \qquad \theta \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}. \tag{1.17}$$

If $p = 1$, the associated characters are periodic (of period u) in the parameter (twist) θ and the theory, which is unitary then, is also minimal as the number of characters is finite. However, if one relaxes the constraint $p = 1$, the characters are no longer periodic in θ. This observation is actually far from being innocuous: it is at the root of the complication in establishing the behaviour of the characters under the modular group. Although non unitary and non minimal, such theories happen to be relevant in the description of non-critical $N = 2$ strings in particular [34–36]. An algebraic structure closely related to the $N = 2$ superconformal algebra, and therefore also instrumental in non critical superstring theory, is the affine Lie superalgebra $\widehat{s\ell}(2|1)$ at level,

$$k = \frac{p}{u} - 1. \tag{1.18}$$

It is precisely for these levels that the superalgebra possesses the so-called 'admissible' representations [37].

In our efforts to master the modular transformations of admissible $N = 2$ and $\widehat{s\ell}(2|1)$ characters, we are led to generalise the theta function theory by studying the modular transformations of functions that are not doubly quasiperiodic in variable(s) belonging to $\mathbb{C}$, and which are called *level-p Appell functions.* For a positive integer p, we define the level-p Appell function as [4]

$$\mathcal{K}_p(\tau,\nu,\mu)=\sum_{m\in\mathbb{Z}}\frac{e^{i\pi m^2 p\tau+2i\pi mp\nu}}{1-e^{2i\pi(\nu+\mu+m\tau)}},\qquad \tau,\ \nu,\mu\in\mathbb{C},\ Im(\tau)>0\ . \tag{1.19}$$

The T and S transformations of $\mathcal{K}_p$ are then as follows,

$$\mathcal{K}_p(\tau+1,\nu,\mu)=\begin{cases}\mathcal{K}_p(\tau,\nu\pm\frac{1}{2},\mu\mp\frac{1}{2}), & p \text{ odd}\ ,\\ \mathcal{K}_p(\tau,\nu,\mu), & p\text{ even}.\end{cases} \tag{1.20}$$

and

$$\begin{aligned}\mathcal{K}_p(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau})&=\tau e^{i\pi p\frac{\nu^2-\mu^2}{\tau}}\,\mathcal{K}_p(\tau,\nu,\mu)\\ &\quad+\tau\sum_{a=0}^{p-1}e^{i\pi p\frac{(\nu+\frac{a}{p}\tau)^2}{\tau}}\,\Phi(p\tau,p\mu-a\tau)\vartheta(p\tau,p\nu+a\tau),\end{aligned} \tag{1.21}$$

where

$$\Phi(\tau,\mu)=-\frac{i}{2\sqrt{-i\tau}}-\frac{1}{2}\int_{\mathbb{R}}dx\,e^{-\pi x^2}\,\frac{\sinh\left(\pi x\sqrt{-i\tau}(1+2\frac{\mu}{\tau})\right)}{\sinh\left(\pi x\sqrt{-i\tau}\right)}. \tag{1.22}$$

Modular transformation properties of theta functions [5] can be considered to underlie the well-known modular group representation on a class of characters of affine Lie algebras [3]. That a modular group representation can be associated with a set of primary fields (φ_A) is often taken as the basic criterion that (φ_A) consistently define a conformal field theory model. Moreover, modular properties of theta functions can be derived from their quasiperiodicity under lattice translations. That the characters $\chi(\tau,\nu,\dots)$ expressed through the theta functions carry a modular group representation, similarly, is intimately related to the fact that they are quasiperiodic under *spectral flow transformations*[6]. A well-known example is provided by the admissible characters of $\widehat{s\ell}(2)$ at level $k=\frac{u}{p}-2$ introduced in Chapter 2. Their modular S-transform may schematically be given as

$$\mathrm{S}.\chi_{r,s,u,p;\theta}=\sum_{r',s',\theta'}S_{(r,s,\theta);(r',s',\theta')}\chi^{\widehat{s\ell}(2)}_{r',s',u,p;\theta'}.$$

[4]Note that for $\mu\to i\infty$ and $p=1$, one recovers the theta function (2.2.3).

[5]See Chapter 2 for a brief review on theta functions.

[6]The name is taken over from the $N=2$ superconformal algebra [9].

However the characters $\chi(\tau, \nu, \ldots)$ that are *not* quasiperiodic in $\nu, \ldots$, (i.e., are not spectral-flow periodic) cannot be rationally expressed through theta functions and do not fit the above pattern. Their modular properties must therefore be different from those of quasiperiodic characters. On the other hand, every consistent conformal field theory model can nevertheless be expected to lead to a "reasonable" modular behaviour of an appropriate set of characters, including those that are not spectral-flow periodic. This raises the question of properly generalising the above pattern to such characters, and a simple object with 'open' quasiperiodicity properties has been known since the nineteenth century: the 'Appell function' $\mathcal{X}_{\tilde{\mu}}(x', y')$ introduced by M.P. Appell [1],

$$\mathcal{X}_{\tilde{\mu}}(x', y') = \frac{i\pi}{2K} \sum_{n\in\mathbb{Z}} e^{\frac{i\pi\tilde{\mu}ny'}{K}} \tilde{q}^{\tilde{\mu}n(n-1)} \frac{e^{\frac{i\pi(x'-y')}{K}} + \tilde{q}^{2n}}{e^{\frac{i\pi(x'-y')}{K}} - \tilde{q}^{2n}}, \tag{1.23}$$

where $\tilde{\mu}$ is a positive integer and $\tilde{q} = e^{-\pi\frac{K'}{K}}$ with K and K', the half-periods of an associated function of the complex variable $z \in \mathbb{C}$. $\mathcal{X}_{\tilde{\mu}}(x', y')$ converges for any $x', y' \in \mathbb{C}$, except when $x' - y' = 2K\mathbb{Z} + 2iK'\mathbb{Z}$.

Appell studied doubly periodic functions of a complex variable z,

$$\begin{aligned} \phi(z + 2K) &= e^{az+b}\phi(z) \\ \phi(z + 2iK') &= e^{a'z+b'}\phi(z), \qquad a, b, a', b' \text{ constants.} \end{aligned}$$

It is always possible to multiply $\phi(z)$ by a phase of the form $e^{\lambda z^2 + \lambda' z}$ so that the resulting function $f(z) = e^{\lambda z^2+\lambda' z}\phi(z)$ is periodic in the K-direction and quasiperiodic in the K'-direction, namely,

$$\begin{aligned} f(z + 2K) &= f(z) \\ f(z + 2iK') &= e^{Az+B}f(z), \qquad \text{with } A = -\frac{mi\pi}{K}, \quad m \in \mathbb{Z}. \end{aligned}$$

If $m \neq 0$, $f(z)$ is said to be *elliptic of the third kind.* If one supposes $f(z)$ is meromorphic (i.e. all singular points at finite distance are poles), the interpretation of the integer m is that it is the difference between the number of zeros and poles the function $f(z)$ possesses in a parallelogram of periods $2K$ and $2iK'$. Appell's work consists in decomposing $f(z)$ as a sum of simple elements (i.e. as a sum of functions having a single pole in the parallelogram of periods) and possibly an integer part. His Appell function shows up in the decomposition for $m < 0$.

We are aware of two recent papers in the mathematical literature which use the Appell function quoted above. Polischuk [7] investigates the geometrical meaning of the Appell function by establishing a connection between vector bundles of rank 2 on elliptic curves and the function

$$\kappa(\tau,\nu,\mu) = \sum_{n\in\mathbb{Z}} \frac{e^{i\pi\tau n^2+2i\pi n\nu}}{e^{2i\pi n\tau} - e^{2i\pi\mu}} \tag{1.24}$$

where $\mu,\nu,\tau \in \mathbb{C}$ and $Im(\tau) > 0$, $\mu \notin \mathbb{Z} + \mathbb{Z}\tau$. It is clear from (1.19) that one has

$$\kappa(\tau,\nu,\mu) = \mathcal{K}_1(\tau,\tau-\nu,-\tau+\nu+\mu)\ . \tag{1.25}$$

On the other hand, setting $K = 1$ and $K' = -i\tau$ and considering $\tilde{\mu} = 1$ in (1.23), we may write

$$\mathcal{X}_1(\tau,\nu,\mu) = \frac{i\pi}{2}\{\sum_{n\in\mathbb{Z}} e^{2i\pi\nu n+2i\pi\tau\frac{n^2-n}{2}} - 2\sum_{n\in\mathbb{Z}} \frac{e^{2i\pi\tau\frac{n^2+n}{2}-2i\pi\nu n}}{1-e^{2i\pi\mu+2i\pi\tau n}}\} \tag{1.26}$$

where $2\mu = x' - y', 2\nu = y'$, or again,

$$\mathcal{X}_1(\tau,\nu+\frac{\tau}{2},\mu) = \frac{i\pi}{2}\vartheta_{0,0}(\tau,\nu) - i\pi\mathcal{K}_1(\tau,-\nu,\mu+\nu), \tag{1.27}$$

where

$$\vartheta_{0,0}(\tau,\nu) = \sum_{n\in\mathbb{Z}} e^{2i\pi\nu n+2i\pi\tau\frac{n^2}{2}}. \tag{1.28}$$

So we see that the κ function of Polischuk differs by a $\vartheta_{0,0}$-term from the original Appell function, and is very closely related to the level one Appell function defined in (1.19) for $p = 1$. It is also closely related to the function used by Kac and Wakimoto in [11] to express the integrable level one (i.e. $k = 1 = u$ in (1.18)) $\widehat{s\ell}(2|1)$ irreducible highest weight representations [7].

[7]The multivariable generalisation of the Appell function, relevant to integrable level one $\widehat{sl(m/1)}$ irreducible characters is quoted in (2.3.4).

But we are interested in another type of generalisation than Kac and Wakimoto's, as we are studying admissible $\widehat{s\ell}(2|1)$ representations. The higher-level Appell functions (1.19) are the sought-after tools, where p is directly related to the parameter p entering the central charge and level formulas (1.1) and (1.18). The functions $\mathcal{K}_{(p)}(q,x,y)$ [8] with $q=e^{2i\pi\tau}, x=e^{2i\pi\nu}$ and $y=e^{2i\pi\mu}$ are quasiperiodic in x and satisfy an *inhomogeneous* finite-difference equations with the inhomogeneous terms given by theta functions, namely

$$\mathcal{K}_{(p)}(q,x,y\,q)=q^{\frac{p}{2}}\,y^p\mathcal{K}_{(p)}(q,x,y)+\sum_{a=0}^{p-1}x^a\,y^a\,q^a\,\theta(q^p,x^p\,q^a). \tag{1.29}$$

We recall the result in [7] that the difference between the Appell function κ and its S transform is divisible by $\vartheta(\tau,\mu)$. The formula (1.21) generalises this to $p>1$ and in addition gives an integral representation for the "kernel" Φ accompanying the theta-functional "additions" to the modular transform. This function, which is an important ingredient of the theory of higher-level Appell functions, can therefore be studied similarly to Barnes-related functions arising elsewhere [16–22].
Remarkably, the theta functions occurring in the right-hand side of (1.21) are those that violate quasiperiodicity of higher-level Appell functions. Open quasiperiodicity in Eq. (1.29) can be recast into an *invariance* statement by considering not the $\mathcal{K}_p$ function alone, but the $(p+1)$-vector $\mathbb{K}_p$ constructed by unifying $\mathcal{K}_p$ with the associated theta functions,

$$\mathbb{K}_p(\tau,\nu,\mu)=\begin{pmatrix}\mathcal{K}_p(\tau,\nu,\mu)\\ \vartheta^{(\mathrm{p})}(\tau,\nu)\end{pmatrix} \tag{1.30}$$

where $\vartheta^{(\mathrm{p})}(\tau,\nu)$ is the p-dimensional vector with components

$$\vartheta_r^{(p)}(\tau,\nu)=e^{2i\pi r\nu}e^{i\pi\frac{r^2}{p}\tau}\vartheta(p\tau,p\nu+r\tau),\qquad 0\le r\le p-1. \tag{1.31}$$

It appears [5] that the vector $\mathbb{K}_p(\tau,\nu,\mu)$ is invariant under the action of the subgroup $\Gamma_{1,2p}$ of $SL(2,\mathbb{Z})$ given below, with a $(p+1)\times(p+1)$ matrix automorphy factor $\mathbf{J}_{\mathrm{p}}(\mathbb{K}_{\mathrm{p}},\gamma;\tau,\nu,\mu)$[9], $\gamma\in\Gamma_{1,2p}$.

[8]We distinguish functions of the variables $\tau,\nu,\mu,...$ from functions of the exponentiated forms $q,x,y,...$ by introducing parentheses around suffices. For instance we use $\mathcal{K}_p(\tau,\nu,\mu)$ and $\mathcal{K}_{(p)}(q,x,y)$. See also (2.3.6). This notation has been also used for character functions.

[9]An explicit expression will be given in [29]

$\Gamma_{1,2p}$ is the subgroup of $SL(2,\mathbb{Z})$ consisting of matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $ab \equiv 0 \bmod 2p$ and $cd \equiv 0 \bmod 2p$. Its action on $\mathcal{H} \times \mathbb{C}^2$ ($\mathcal{H}$ being the upper-half plane) is given by [10]:

$$\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, \nu, \mu) \to (\gamma\tau, \gamma\nu, \gamma\mu) = (\frac{a\tau + b}{c\tau + d}, \frac{\nu}{c\tau + d}, \frac{\mu}{c\tau + d}), \tag{1.32}$$

while its action on functions $f : \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}^{p+1}$ is given by,

$$\gamma \cdot f(\tau, \nu, \mu) = \mathbf{J}_p(f, \gamma; \tau, \nu, \mu)\, f(\gamma\tau, \gamma\nu, \gamma\mu), \tag{1.33}$$

where $\mathbf{J}_p(f, \gamma; \tau, \nu, \mu)$ is a $(p+1) \times (p+1)$ matrix automorphy factor. By invariance of $f(\tau, \nu, \mu)$ under $\Gamma_{1,2p}$, we mean that

$$\gamma \cdot f(\tau, \nu, \mu) = f(\tau, \nu, \mu), \qquad \forall \gamma \in \Gamma_{1,2p}. \tag{1.34}$$

Hence, the invariance statement on the $p+1$-vector $\mathbb{K}_p(\tau, \nu, \mu)$ is the following,

$$\gamma \cdot \mathbb{K}_p(\tau, \nu, \mu) = \mathbf{J}_p(\mathbb{K}_p, \gamma; \tau, \nu, \mu)\, \mathbb{K}_p(\gamma\tau, \gamma\nu, \gamma\mu) = \mathbb{K}_p(\tau, \nu, \mu), \qquad \forall \gamma \in \Gamma_{1,2p}. \tag{1.35}$$

An interesting question is whether characters expressible through differences of higher-level Appell functions exhibit modular properties whose structure is similar to that of higher-level Appell functions themselves, i.e. (1.20)–(1.21). Although a more thorough analysis should be carried out, the results obtained in this thesis on the S transform of $N = 2$ and $\widehat{s\ell}(2|1)$ characters pertaining to the classes discussed above (see (4.2.75) and (5.2.36)) point to the following structure. Consider the $(2pu+1)$-vectors $\mathbb{W}_{r,s,u,p;\theta}(\tau, \nu)$ (resp. $\mathbb{X}_{r,s,u,p;\theta}(\tau, \nu, \mu)$) constructed by unifying each $N = 2$ (resp. $\widehat{s\ell}(2|1)$) admissible character $\omega_{r,s,u,p;\theta}(\tau, \nu)$ (resp. $\chi_{r,s,u,p;\theta}(\tau, \nu, \mu)$), with the $2pu$ functions $\Lambda_{r,s,u,p}(\tau, 0)$ (resp. $\Lambda_{r,s,u,p}(\tau, \nu)$) defined in (C.2.1) and arising in the corresponding open quasiperiodicity formulas (4.1.11). Since $0 \le r \le u-1$, $1-p \le s \le p$, $0 \le \theta \le u-1$, there are $2pu^2$ such vectors and we define

$$\mathbb{W}_{r,s,u,p;\theta}(\tau, \nu) = \begin{pmatrix} \omega_{r,s,u,p;\theta}(\tau, \nu) \\ \mathbf{\Lambda}^{(\mathbf{2pu})}(\tau, 0) \end{pmatrix}, \qquad \mathbb{X}_{r,s,u,p;\theta}(\tau, \nu, \mu) = \begin{pmatrix} \chi_{r,s,u,p;\theta}(\tau, \nu, \mu) \\ \mathbf{\Lambda}^{(\mathbf{2pu})}(\tau, \nu) \end{pmatrix} \tag{1.36}$$

[10]The notation $\gamma\nu$ and $\gamma\mu$ is somewhat lose, because the action of $\Gamma_{1,2p}$ on ν and μ depends on τ.

where $\mathbf{\Lambda}^{(\mathbf{2pu})}(\tau,\nu)$ is the $2pu$-dimensional vector with components $\Lambda_{r,s,u,p}(\tau,\nu)$. Then the action of the modular transformation S closes on each set of $2pu^2$ vectors $\mathbb{W}_{r,s,u,p;\theta}(\tau,\nu)$ and $\mathbb{X}_{r,s,u,p;\theta}(\tau,\nu,\mu)$.

The layout of the thesis is as follows. Chapter 2 sets out the notations and definitions as well as the basic properties of theta and higher-level Appell functions. Chapter 3 deals with the derivation of the modular properties of higher-level Appell functions. As we explained before, the S modular transform of Appell functions yields an important function of two variables, namely $\Phi(\tau,\nu)$, which we study thoroughly at the end of the chapter. In chapter 4, we consider admissible $N=2$ characters and rewrite them in terms of higher-level Appell functions in order to study their modular behaviour. Even though we have control on how the Appell functions S-transform, the derivation of the S-transform remains highly non-trivial as characters are written as differences of Appell functions, and one must reconstruct those differences after the S transformation in order to re-express the transformed characters in terms of $N=2$ characters again, modulo corrective terms given in terms of ϑ and Φ functions. We are able to perform an important consistency check of our formulas at the end of this chapter in the case where $p=1$. Indeed, the $N=2$ characters become minimal and unitary in this instance, and are expressible as ratios of ϑ-type functions. Hence their S-transform is relatively 'easy' and has been known for decades. However, we re-calculate it in an independent way, and show that it yields the same result as the one obtained by setting $p=1$ in the general S-transformation. Chapter 5 analyses the case of admissible $\widehat{s\ell}(2|1)$ characters along similar lines as the admissible $N=2$ characters. The structure of S-transforms falls in the pattern described earlier. Finally, we make another consistency check in Chapter 6. It has been known for a while that the admissible $N=2$ characters may be obtained by taking the residue of admissible $\widehat{s\ell}(2|1)$ characters at a point $x=q^n,\ \ n\in\mathbb{Z}/\{u\mathbb{Z}+s-1\}$. We therefore show how to induce the behaviour of admissible $N=2$ characters under the modular group from that of admissible $\widehat{s\ell}(2|1)$ characters. We finally summarise our results in the conclusions.

Chapter 2

Theta functions and Appell functions properties

2.1 Introduction

In this chapter we start by recalling the definition of theta functions and we introduce our notations as well as some properties of these functions which will be relevant to our work. We then introduce Appell functions, including many of their remarkable properties. The quasi- and open quasi-periodicity properties in particular will be used in an attempt to formulate the modular transformation properties of more elaborate functions, whose building blocks are the Appell functions.

Period increasing statements for both theta functions and more widely for higher-level Appell functions are discussed as a separate section. Furthermore some remarkable technical relations between higher-level Appell functions and theta functions are worked out at the end of this chapter.

It is worth noting that, apart from some very recent geometrical developments, higher-level Appell functions properties have not appeared in the literature to our knowledge, and we are presenting explicit proofs in the appropriate sections or appendices, or we have at least given an efficient clue to derive the formulas as easily as possible.

2.2 Theta functions

Theta functions appear in many different contexts in the mathematical and physical literature, mainly because they may be used to construct doubly periodic meromorphic [1] functions on the complex plane, also called *elliptic functions.* They will play an important role in this thesis, not only because they serve as building blocks for the construction of characters of a large class of representations of infinite-dimensional Lie (super)algebras and (super)conformal algebras, but they also appear in the description of other classes of characters (notably *admissible* characters) of these algebras.

A function $T : \mathbb{C} \to \mathbb{C} : \nu \to T(\nu)$ is called a *theta function with quasiperiods* 1 *and* τ, *and characteristic* $(a_1, b_1; a_2, b_2)$ if,

$$T(\nu+1) = e^{a_1\nu+b_1}T(\nu), \qquad T(\nu+\tau) = e^{a_2\nu+b_2}T(\nu). \tag{2.2.1}$$

So theta functions are *quasi-doubly periodic functions in the complex variable* ν. The parallelogram of periods (see Figure 3.1), with opposite sides identified is a torus of modulus τ. In particular, if T is a theta function with quasiperiods 1 and τ, then $E = (\frac{T'}{T})'$ is an elliptic function with periods 1 and τ, i.e.

$$E(\nu+1) = E(\nu), \qquad E(\nu+\tau) = E(\nu). \tag{2.2.2}$$

The *degree* of a theta function is given by $\frac{1}{2i\pi}(a_1\tau - a_2)$, and it can be shown that it is always an integer.

The following theta function of degree 1 and quasiperiods 1 and τ is central to our work,

$$\vartheta(\tau,\nu) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n\nu}\,, \tag{2.2.3}$$

where $\nu \in \mathbb{C}$ and τ is a complex parameter with $\mathrm{Im}(\tau) > 0$. One can easily check the following quasiperiodicity properties of $\vartheta(\tau,\nu)$ in the variable ν,

$$\begin{aligned} \vartheta(\tau,\nu+k) &= \vartheta(\tau,\nu) \qquad k \in \mathbb{Z}\,, && (2.2.4)\\ \vartheta(\tau,\nu+\theta\tau) &= e^{-\pi i(\theta^2\tau+2\theta\nu)}\vartheta(\tau,\nu) \qquad \theta \in \mathbb{Z}. && (2.2.5) \end{aligned}$$

[1] A meromorphic function is a function whose sole singularities in the complex plane are poles.

We also define,

$$\vartheta(\tau,\nu)=\theta(e^{2\pi i\tau},e^{2\pi i\nu})=\theta(q,z)\ , \tag{2.2.6}$$

and re-express the second quasiperiodic behaviour (2.2.5) in the variables $q=e^{2\pi i\tau}$ and $z=e^{2\pi i\nu}$ for future reference,

$$\theta(q,zq^{\theta})=q^{-\theta^2/2}z^{-\theta}\theta(q,z)\ . \tag{2.2.7}$$

The function (2.2.3) has one unique zero in the parallelogram of periods in the ν-plane at $\nu=\frac{1}{2}(1+\tau)$. Using this fact as well as (2.2.6), the formula (2.2.3) can be also expressed as the following infinite product,

$$\theta(q,z)=\prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-\frac{1}{2}})(1+z^{-1}q^{m-\frac{1}{2}})\,. \tag{2.2.8}$$

In this thesis, we often refer to the following functions (note that $\vartheta_{(1,1)}$ and $\vartheta_{(1,0)}$ are not theta functions according to the definition above, but we will however allow ourselves to call them theta functions),

$$\begin{aligned}
\vartheta_{(1,1)}(q,z) &= \theta(q,-zq^{\frac{1}{2}})=\sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{1}{2}(m^2-m)}z^{-m}\\
&= \prod_{m\geq 0}(1-z^{-1}q^m)\prod_{m\geq 1}(1-zq^m)\prod_{m\geq 1}(1-q^m)\\
&= iz^{-\frac{1}{2}}q^{-\frac{1}{8}}\vartheta_{(1)}(q,z) && (2.2.9)\\
\vartheta_{(1,0)}(q,z) &= \theta(q,zq^{\frac{1}{2}})=\sum_{m\in\mathbb{Z}}q^{\frac{1}{2}(m^2-m)}z^{-m}\\
&= \prod_{m\geq 0}(1+z^{-1}q^m)\prod_{m\geq 1}(1+zq^m)\prod_{m\geq 1}(1-q^m)\\
&= z^{-\frac{1}{2}}q^{-\frac{1}{8}}\vartheta_{(2)}(q,z) && (2.2.10)\\
\vartheta_{(0,0)}(q,z) &= \theta(q,z)=\sum_{m\in\mathbb{Z}}q^{\frac{1}{2}m^2}z^{-m}\\
&= \vartheta_{(3)}(q,z^{-1}) && (2.2.11)\\
\vartheta_{(0,1)}(q,z) &= \theta(q,-z)=\sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{1}{2}m^2}z^{-m}\\
&= \vartheta_{(4)}(q,z^{-1})\ . && (2.2.12)
\end{aligned}$$

In the above, $\vartheta_{(i)}(q,z), i=1,2,3,4$ are the *Jacobi theta functions*[2].

The quasiperiodicity of the above functions under $\nu \to \nu+\theta\tau$ for $\theta\in\mathbb{Z}$ is expressed as,

$$\begin{aligned}\vartheta_{(1,1)}(q,zq^{\theta}) &= (-1)^{\theta}q^{-\frac{1}{2}(\theta^2+\theta)}z^{-\theta}\vartheta_{(1,1)}(q,z), &(2.2.13)\\ \vartheta_{(1,0)}(q,zq^{\theta}) &= q^{-\frac{1}{2}(\theta^2+\theta)}z^{-\theta}\vartheta_{(1,0)}(q,z), &(2.2.14)\\ \vartheta_{(0,0)}(q,zq^{\theta}) &= q^{-\frac{1}{2}\theta^2}z^{-\theta}\vartheta_{(0,0)}(q,z) &(2.2.15)\\ \vartheta_{(0,1)}(q,zq^{\theta}) &= (-1)^{\theta}q^{-\frac{1}{2}\theta^2}z^{-\theta}\vartheta_{(0,1)}(q,z)\ . &(2.2.16)\end{aligned}$$

We also recall here the definition of the eta function as it was originally introduced by Dedekind [13] ,

$$\eta(\tau)=\tilde{\eta}(q)=q^{\frac{1}{24}}\sum_{m=0}^{\infty}(-1)^m q^{\frac{1}{2}(3m^2+m)}=q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m)\,. \qquad (2.2.17)$$

Now based on what we explained in the introductory chapter, the S modular transformation of the above theta functions are given by,

$$\begin{aligned}\vartheta_{1,1}(-\frac{1}{\tau},\frac{\nu}{\tau}) &= -i\sqrt{-i\tau}\,e^{i\pi\nu+i\pi\frac{(\nu-\frac{1}{2})^2}{\tau}+\frac{i\pi}{4}\tau}\vartheta_{1,1}(\tau,\nu), &(2.2.18)\\ \vartheta_{1,0}(-\frac{1}{\tau},\frac{\nu}{\tau}) &= \sqrt{-i\tau}\,e^{i\pi\frac{(\nu-\frac{1}{2})^2}{\tau}}\vartheta_{1,0}(\tau,\frac{1}{2}+\nu-\frac{\tau}{2}), &(2.2.19)\end{aligned}$$

and

$$\vartheta(-\tfrac{1}{\tau},\tfrac{\nu}{\tau})=\sqrt{-i\tau}\,e^{i\pi\frac{\nu^2}{\tau}}\vartheta(\tau,\nu)\ . \qquad (2.2.20)$$

Their transformations under T simply are,

$$\begin{aligned}\vartheta_{1,1}(\tau+1,\nu) &= \vartheta_{1,1}(\tau,\nu), &(2.2.21)\\ \vartheta_{1,0}(\tau+1,\nu) &= \vartheta_{1,0}(\tau,\nu), &(2.2.22)\\ \vartheta(\tau+1,\nu) &= \vartheta_{1,1}(\tau,\nu-\frac{\tau}{2}). &(2.2.23)\end{aligned}$$

Finally, both these transformations act on the eta function as,

$$\eta(\tau+1)=e^{\frac{i\pi}{12}}\eta(\tau),\qquad \eta(-\tfrac{1}{\tau})=\sqrt{-i\tau}\,\eta(\tau)\ . \qquad (2.2.24)$$

[2]After formula (2.2.16) we don't mention about $\vartheta_{(01)}$ any more, as it will not be used in this thesis from now on.

Modular transformation properties of theta functions can be considered to underlie the well-known modular group representation on a class of characters of affine Lie algebras [3]. Moreover, modular properties of theta functions can be derived from their quasiperiodicity under lattice translations. That the affine characters $\chi(\tau,\nu,..)$ expressed through the theta functions carry a modular group representation is similarly rooted in the fact that they are quasiperiodic under *spectral flow transformations*[3] One therefore has the following diagrammatic implications:

$$\begin{array}{ccc} \text{quasiperiodicity of theta functions} & \longrightarrow & \text{spectral-flow periodicity of characters} \\ \downarrow & & \downarrow \\ \text{modular properties of theta functions} & \longrightarrow & \text{modular group representation on characters} \end{array}$$

Let us take the well-known example of the affine Lie algebra $\widehat{s\ell}(2)$ at fractional level $k=\frac{u}{p}-2$, with $\frac{u}{p}>0$ and u and p coprime [3,6]. The corresponding *admissible* (untwisted) characters are labelled by two integers r and s in the ranges $1\le r\le u-1$ and $1\le s\le p$. They are given by,

$$\chi^{\widehat{s\ell}(2)}_{(r,s,u,p)}(q,z)=z^{\frac{r-1}{2}-(s-1)\frac{u}{2p}}q^{\frac{pr^2}{4u}-\frac{r}{2}(s-1)+\frac{u(s-1)^2}{4p}-\frac{up}{4}}$$
$$\frac{\vartheta_{(1,0)}(q^{2up},z^u q^{p(r-u)-(s-1)u})-z^{-r}q^{r(s-1)}\vartheta_{(1,0)}(q^{2up},z^u q^{-p(r+u)-(s-1)u})}{\vartheta_{(1,1)}(q,z)}\ . \qquad (2.2.25)$$

These character functions are quasiperiodic under the spectral flow $z\to zq^{2p}$, except for the character $\chi_{u/2,s,u,p}(q,z)$ when u is even, which is quasiperiodic under $z\to zq^p$. Indeed, one finds that for $\theta\in\mathbb{Z}$,

$$\chi^{\widehat{s\ell}(2)}_{(r,s,u,p)}(q,zq^\theta)=(-1)^\theta q^{-\frac{k}{4}\theta^2}z^{-\frac{k}{2}\theta}\chi^{\widehat{s\ell}(2)}_{(r,s-\theta,u,p)}(q,z), \qquad (2.2.26)$$

and in particular,

$$\begin{aligned}\chi^{\widehat{s\ell}(2)}_{(r,s,u,p)}(q,zq^p) &= (-1)^p q^{-\frac{k}{4}p^2}z^{-\frac{k}{2}p}\chi^{\widehat{s\ell}(2)}_{(r,s-p,u,p)}(q,z)=(-1)^{p+1}q^{-\frac{k}{4}p^2}z^{-\frac{k}{2}p}\chi^{\widehat{s\ell}(2)}_{(u-r,s,u,p)}(q,z)\\ \chi^{\widehat{s\ell}(2)}_{(r,s,u,p)}(q,zq^{2p}) &= q^{-kp^2}z^{-kp}\chi^{\widehat{s\ell}(2)}_{(r,s,u,p)}(q,z)\ . \qquad (2.2.27)\end{aligned}$$

[3]The name is taken from the $N=2$ superconformal algebra.

These quasiperiodicity properties ensure that the set of $\widehat{s\ell}(2)$ admissible characters is closed and finite under spectral flow. Furthermore, it is precisely this finite set of characters that carries a representation of the modular group. Schematically, one finds their transformation under S to be,

$$S.\chi_{r,s,u,p} = \sum_{r',s'} S_{(r,s);(r',s')}\chi_{r',s',u,p} \tag{2.2.28}$$

Quasiperiodicity of characters under spectral flow therefore ensures they carry a finite-dimensional representation of the modular group. This is also the case for the unitary representations of the $N = 2$ superconformal algebra, which is discussed in Chapter 4.

But the characters $\chi(\tau, \nu, ...)$ that are *not* quasiperiodic in $\nu, ...$ cannot be rationally expressed through theta functions and do not fit the above pattern. Their modular properties must therefore be different from those of quasiperiodic characters, and it is an interesting mathematical question to ask how to generalise the above pattern to certain classes of characters which obey *open quasiperiodic properties* under spectral flow. (Note that the terminology 'additive-quasiperiodic properties' has been used somewhere else [8].) We will extensively study two particular classes of such characters later: the non-unitary admissible $N = 2$ superconformal [9] characters and the admissible affine $\widehat{s\ell}(2|1)$ characters. We are therefore seeking 'basic' objects which generalise the theta functions encountered here and which possess good modular transformation properties, such that they lead to reasonable modular transformation properties of the characters, namely,

$$S.\chi_{r,s,u,p;\theta} = \sum_{r',s',\theta'} S_{(r,s,\theta);(r',s',\theta')}\chi_{r',s',u,p;\theta'} \tag{2.2.29}$$

A simple object of the sought type, including all essential open quasiperiodic behaviours, is introduced now under the name of Appell function.

2.3 Higher-level Appell functions

Historically M.A. Appell introduced his 'Appell function' [4]in his mathematics article [1] about doubly-periodic functions . It was then used in a wide range of algebraic and geometrical contexts afterwards, with some minor changes of definition to adapt its use to a variety of contexts. For example in Polischuk's paper on elliptic curves [7] the Appell function appears in the following form,

$$\kappa(Y,X,\tau)=\sum_{n\in\mathbb{Z}}\frac{e^{i\pi n^2\tau}e^{2i\pi nX}}{e^{2i\pi n\tau}-e^{2i\pi Y}}\ , \tag{2.3.1}$$

where $Y, X, \tau\ \in \mathbb{C}$ and $Im\tau > 0$, $Y \notin \mathbb{Z}+\tau\mathbb{Z}$. This differs by a theta function from Appell's definition. In fact Polischuk's notations are more similar to Halphen's book [10] than to Hermite's or even Appell's ones.

Another famous and outstanding definition was introduced by Kac and Wakimoto in [11]. It reads,

$$A(a,z,q)=\sum_{k\in\mathbb{Z}}\frac{q^{\frac{1}{2}k^2}z^k}{1+aq^k}\ , \tag{2.3.2}$$

and converges to a meromorphic function in the domain $a, z, q \in \mathbb{C}$, $\mid q \mid< 1$. In the case of $a = 0$, one recovers the usual theta function (2.2.3). It can be also linked to an Appell function κ in the following way,

$$A(-y,x^{-1}q,q)=\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n^2}{2}+n}x^{-n}}{1-yq^n}=\kappa(y,x,q)\ , \tag{2.3.3}$$

where $y = e^{2i\pi Y}$, $x = e^{2i\pi X}$ and $q = e^{2i\pi\tau}$. It is worth noting that a multivariable form of (2.3.2) has been also used in [11], namely,

$$A_{B,l}(a;z_1,...,z_n;q)=\sum_{k\in\mathbb{Z}^N}\frac{q^{\frac{1}{2}k^TBk}z_1^{k_1}...z_N^{k_N}}{1+aq^{l(k)}}\ . \tag{2.3.4}$$

B is an $N\times N$ symmetric matrix such that $Re(B)$ is positive definite and $l(k)$ is a linear function of $\mathbb{C}^N$. Here, putting $a = 0$ gives us the theta function in its multivariable form [14]. However we believe using a non-multivariable Appell function

[4]The same function was introduced by M.Hermite essentially for the same problem, however, he did not publish his results until the appearance of the first part of Appell's paper.

has always been good enough for our work. For the study of modular transformations of non-unitary $N = 2$ and $\widehat{s\ell}(2|1)$ characters, we consider a generalisation of Appell's definition, namely higher-level Appell functions represented by $\mathcal{K}_{(p)}$ for a level $p \in \mathbb{N}$ in what follows [5].

For any positive integer p, **the level p Appell function** $\mathcal{K}_{(p)}$ is,

$$\mathcal{K}_{(p)}(q,x,y) = \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m^2 p}{2}} x^{mp}}{1 - xyq^m} , \tag{2.3.5}$$

where $x = e^{2i\pi\nu}, y = e^{2i\pi\mu}, q = e^{2i\pi\tau}$ with $\nu, \mu, \tau \in \mathbb{C}$ and $Im\,\tau > 0$.

We also define,

$$\begin{aligned} \mathcal{K}_p(\tau,\nu,\mu) &= \mathcal{K}_{(p)}(e^{2i\pi\tau}, e^{2i\pi\nu}, e^{2i\pi\mu}) \\ &= \mathcal{K}_{(p)}(q,x,y) , \end{aligned} \tag{2.3.6}$$

and use both representations $\mathcal{K}_p$ and $\mathcal{K}_{(p)}$ all throught the current thesis.

Note that unlike the theta functions introduced in the previous section, the higher-level Appell functions defined above have singularities whenever $\nu + \mu + m\tau = k$ for $m, k \in \mathbb{Z}$. These functions will be of a great benefit for our calculations in Chapters 4 and 5 and can be related to all former existing definitions of Appell functions. For instance (2.3.3) may be rewritten as,

$$\mathcal{K}_{(1)}(q, y^{-1}q, xyq^{-1}) = \kappa(x,y,q) . \tag{2.3.7}$$

We now examine the periodicity properties of higher-level Appell functions.

2.4 Properties of higher-level Appell functions

A difficult job for us in fact was, guessing, extracting and fully deriving many properties of the higher-level Appell functions with a view to use them for our further

[5]Incidentally, a certain version of the $\mathcal{K}_0$ function, namely, the function $P_\lambda(x,q) = \mathcal{K}_{(p)}(q, x^{\frac{1}{p}}, \lambda x^{-1}) \mid_{p=0}$ is related to an Eisenstein series and was introduced in [12] for λ a root of unity.

calculations. We hereby classify them in different subsections and explain each property separately.

2.4.1 Basic properties

The function presented in (2.3.5) can be represented also as the following double-series formula,

$$\mathcal{K}_{(p)}(q,x,y) = \left(\sum_{m\geq 0}\sum_{n\geq 0} - \sum_{m\leq -1}\sum_{n\leq -1}\right) q^{\frac{m^2 p}{2}+mn} x^{mp+n} y^n \,, \tag{2.4.1}$$

valid for $|q| < |xy| < 1$.

We note the property,

$$\mathcal{K}_{(p)}(q,x,y) = -y^{-1}x^{-1}\mathcal{K}_{(p)}(q, x^{-1}q^{\frac{1}{p}}, y^{-1}q^{-\frac{1}{p}}) \,. \tag{2.4.2}$$

We also have the following properties, easily derived in the exponential notation,

$$\mathcal{K}_p(\tau,\nu+m,\mu) = \mathcal{K}_p(\tau,\nu,\mu) = \mathcal{K}_p(\tau,\nu,\mu+m), \quad m\in\mathbb{Z}, \tag{2.4.3}$$

$$\mathcal{K}_p(\tau,\nu+\tfrac{m}{p},\mu-\tfrac{m}{p}) = \mathcal{K}_p(\tau,\nu,\mu), \qquad m\in\mathbb{Z}. \tag{2.4.4}$$

Appell functions *of even level* are central to our calculations. They satisfy two elementary identities, namely,

$$\sum_{b=0}^{p-1}\mathcal{K}_{2p}(\tau,\nu,\mu\pm\tfrac{b}{p}) = \sum_{b=0}^{p-1}\mathcal{K}_{2p}(\tau,\nu\pm\tfrac{b}{p},\mu) = p\mathcal{K}_2(p\tau,p\nu,p\mu), \tag{2.4.5}$$

and

$$\mathcal{K}_{2p}(\tau,\nu\pm\frac{m}{2},\mu-\frac{m}{2}) = \mathcal{K}_{2p}(\tau,\nu,\mu), \qquad m\in\mathbb{Z}\,. \tag{2.4.6}$$

2.4.2 Periodicity properties

Similarly to what we saw in the case of theta functions (2.2.13)-(2.2.7), higher-level Appell functions are quasiperiodic in their second argument,

$$\mathcal{K}_{(p)}(q,xq^n,y) = q^{-\frac{n^2p}{2}}x^{-np}\mathcal{K}_{(p)}(q,x,y), \quad n\in\mathbb{Z}. \tag{2.4.7}$$

In addition they also possess an important open quasiperiodicity in their third argument, namely *higher-level Appell functions satisfy the inhomogeneous finite-difference equation where the inhomogeneous terms are theta functions,*

$$\mathcal{K}_{(p)}(q,x,y\,q) = q^{\frac{p}{2}}\,y^p\mathcal{K}_{(p)}(q,x,y) + \sum_{a=0}^{p-1} x^a\,y^a\,q^a\,\theta(q^p,x^p\,q^a). \tag{2.4.8}$$

The above property can be then generalized to,

$$\mathcal{K}_{(p)}(q,x,yq^n) = q^{\frac{n^2p}{2}} y^{np}\mathcal{K}_{(p)}(q,x,y) + \begin{cases} \sum_{j=0}^{pn-1} x^j y^j q^{nj}\theta(q^p, x^p q^j), & n \in \mathbb{N}, \\ -\sum_{j=pn}^{-1} x^j y^j q^{nj}\theta(q^p, x^p q^j), & n \in -\mathbb{N}. \end{cases} \tag{2.4.9}$$

The periodicity properties (2.4.7) and (2.4.9) are proven in Appendix A.1. The property above may be reformulated with only p distinct theta functions in the right-hand side,

$$\begin{aligned} \mathcal{K}_{(p)}(q,x,yq^n) &= q^{\frac{n^2p}{2}} y^{np}\mathcal{K}_{(p)}(q,x,y) \\ &+ \begin{cases} \sum_{j=0}^{n-1} q^{\frac{j(2n-j)p}{2}} y^{jp} \sum_{r=0}^{p-1} x^r y^r q^{(n-j)r}\theta(q^p, x^p q^r), & n \in \mathbb{N}, \\ -\sum_{j=n}^{-1} q^{\frac{j(2n-j)p}{2}} y^{jp} \sum_{r=0}^{p-1} x^r y^r q^{(n-j)r}\theta(q^p, x^p q^r), & n \in -\mathbb{N}. \end{cases} \end{aligned} \tag{2.4.10}$$

And finally another and possibly the most remarkable open quasiperiodic formula for this function is given by,

$$\begin{aligned} &\mathcal{K}_{(p)}(q, xq^{-\frac{n}{p}}, yq^{\frac{n}{p}}) = \\ &= (xy)^n\mathcal{K}_{(p)}(q,x,y) + \begin{cases} \sum_{r=1}^{n} (xy)^{n-r}\theta(q^p, x^p q^{-r}), & n \in \mathbb{N}, \\ -\sum_{r=n+1}^{0} (xy)^{n-r}\theta(q^p, x^p q^{-r}), & n \in -\mathbb{N}. \end{cases} \end{aligned} \tag{2.4.11}$$

2.4.3 Period increasing statements

The theta function $\vartheta(\tau,\nu)$ has quasiperiod τ as seen from (2.2.5). The trivial manipulation

$$\sum_{m\in\mathbb{Z}} \sum_{r=0}^{u-1} f(mu \pm r) = \sum_{m\in\mathbb{Z}} f(m), \tag{2.4.12}$$

allows us to re-express $\theta(q,z)$ as a finite sum of theta functions with increased quasiperiods $u^2\tau$. Indeed, we have

$$\begin{aligned}
\theta(q,z) &= \sum_{m\in\mathbb{Z}} q^{m^2/2} z^m \\
&= \sum_{m\in\mathbb{Z}} \sum_{r=0}^{u-1} q^{\frac{1}{2}(mu+r)^2} z^{(mu+r)} \\
&= \sum_{r=0}^{u-1} q^{r^2/2} z^r \sum_{m\in\mathbb{Z}} q^{m^2u^2/2+rmu} z^{mu} .
\end{aligned}$$

Now substituting the second summation in the formula above with a *period increased* theta function, the infinite summation will be absorbed and we finally arrive at,

$$\theta(q,z) = \sum_{r=0}^{u-1} z^r q^{r^2/2} \theta(q^{u^2}, z^u q^{ru}) . \tag{2.4.13}$$

A similar type of formula for higher-level Appell functions will be extremely useful later, as can be anticipated from noticing the summation-like structure shown in (2.2.28) and (2.2.29) when expressing the modular transformations of $N=2$ and $\widehat{s\ell}(2|1)$ characters. We therefore derive such a 'period increasing' formula. Start with

$$\begin{aligned}
\mathcal{K}_{(p)}(q,x,y) &= \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m^2p}{2}} x^{mp}}{1-xyq^m} \\
&= \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m^2p}{2}} x^{mp}}{1-(xyq^m)^u} \cdot \frac{1-(xyq^m)^u}{1-xyq^m} \\
&= \sum_{m\in\mathbb{Z}} q^{\frac{m^2p}{2}} x^{mp} \sum_{b=0}^{u-1} \frac{(xyq^m)^b}{1-(xyq^m)^u} ,
\end{aligned}$$

where in the second line the following elementary lemma for $A = xyq^m$ has been used,

$$\sum_{b=0}^{u-1} A^b = \frac{1-A^u}{1-A} \qquad A\in\mathbb{C},\ u\geq 1 . \tag{2.4.14}$$

Although in analogy with (2.4.13) one could expect to see a period-increased Appell function at this point , there is no obvious Appell function to be traced there. But using (2.4.12) to enter an extra summation one obtains,

$$\begin{aligned}
\mathcal{K}_{(p)}(q,x,y) &= \sum_{m\in\mathbb{Z}}\sum_{b=0}^{u-1}\sum_{a=0}^{u-1}\frac{q^{\frac{1}{2}p(mu+a)^2+b(mu+a)}x^{p(mu+a)}(xy)^b}{1-(xyq^{(mu+a)})^u} \\
&= \sum_{m\in\mathbb{Z}}\sum_{b=0}^{u-1}\sum_{a=0}^{u-1}\frac{\left[q^{\frac{1}{2}m^2u^2p}(x^uq^{au+\frac{bu}{p}})^{mp}\right]q^{\frac{1}{2}a^2p+ab}x^{ap+b}y^b}{1-x^uy^uq^{(mu^2+au)}} \\
&= \sum_{b=0}^{u-1}\sum_{a=0}^{u-1}q^{\frac{1}{2}a^2p+ab}x^{ap+b}y^b\sum_{m\in\mathbb{Z}}\frac{q^{\frac{1}{2}m^2u^2p}(x^uq^{au+\frac{bu}{p}})^{mp}}{1-[x^uq^{au+\frac{bu}{p}}][y^uq^{-\frac{bu}{p}}]q^{mu^2}}\,.
\end{aligned}$$

So the last summand forms an Appell function-like structure and by substituting that with a new *period-increased* $\mathcal{K}_p$ we obtain,

$$\mathcal{K}_{(p)}(q,x,y)=\sum_{a=0}^{u-1}\sum_{b=0}^{u-1}q^{\frac{1}{2}a^2p+ab}x^{ap+b}y^b\mathcal{K}_{(p)}(q^{u^2},x^uq^{au+\frac{bu}{p}},y^uq^{-\frac{bu}{p}}),\quad u\in\mathbb{N}\,. \tag{2.4.15}$$

Guided by (2.2.29), it much looks like what we need for our S modular transformations investigations. However the formula is still not quite perfect, as a single but vital condition, namely the coprimity of the two numbers u and p, has not been considered yet. We can implement this requirement in two ways, one of which is brought in Appendix A.2. We present the second derivation in the text as it considers higher-level Appell functions of *even level* $2p$ that are certainly the most relevant case to us. So we start with,

$$\begin{aligned}
\mathcal{K}_{(2p)}(q,x,y) &= \sum_{m\in\mathbb{Z}}\frac{q^{pm^2}x^{2mp}}{1-xyq^m} \\
&= \sum_{m\in\mathbb{Z}}q^{m^2p}x^{2mp}\Big[\frac{1-(xyq^m)^{pu}}{[1-(xyq^m)^p][1-(xyq^m)^u]} \\
&+ \sum_{\substack{r=1\\ ur-sp>0}}^{p-1}\sum_{s=1}^{u-1}(xyq^m)^{ur-sp}\Big]\,,
\end{aligned} \tag{2.4.16}$$

in which the following identity, proved in appendix (A.2) has been used

$$\frac{1-q^{pu}}{(1-q^p)(1-q^u)}-\frac{1}{1-q}=-\sum_{\substack{r=1\\ ur-sp>0}}^{p-1}\sum_{s=1}^{u-1}q^{ur-sp}\qquad u,p\in\mathbb{N},\quad (u,p)=1, \tag{2.4.17}$$

modulo the change $q \to (xyq^m)$. Thus from (2.4.16) we continue as,

$$\mathcal{K}_{(2p)}(q,x,y) = \sum_{m\in\mathbb{Z}} \frac{q^{m^2p}x^{2mp}}{1-x^uy^uq^{mu}} \cdot \left[\frac{1-x^{pu}y^{pu}q^{mpu}}{1-x^py^pq^{mp}}\right] + \sum_{\substack{r=1 \\ ur-ps>0}}^{p-1}\sum_{s=1}^{u-1} x^{ur-ps}y^{ur-ps}\theta(q^{2p},x^{2p}q^{ur-ps}) \ , \qquad (2.4.18)$$

where, in the second term, we have used the definition of theta function (2.2.3). Then we apply the lemma (2.4.14) to the first term and get, for this first term,

$$\sum_{m\in\mathbb{Z}} \frac{q^{m^2p}x^{2mp}}{1-x^uy^uq^{mu}} \sum_{s'=0}^{u-1} x^{ps'}y^{ps'}q^{pms'}$$
$$= \sum_{m\in\mathbb{Z}}\sum_{s'=0}^{u-1}\sum_{b=0}^{u-1} \frac{x^{ps'}y^{ps'}q^{p(um-b)^2+p(um-b)s'}x^{2p(um-b)}}{1-x^uy^uq^{u(um-b)}} \ , \qquad (2.4.19)$$

where in the right hand side, the formula (2.4.12) is an essential tool in creating the extra summations (and obviously indices) needed. Now (2.4.19) can be followed as,

$$\sum_{m\in\mathbb{Z}}\sum_{s'=0}^{u-1}\sum_{b=0}^{u-1} x^{ps'}y^{ps'}q^{p(um-b)^2+p(um-b)s'}x^{2p(um-b)}\frac{x^{2pb}y^{2pb}q^{2pumb-2pb^2}}{1-x^uy^uq^{u(um-b)}}$$
$$-\sum_{m\in\mathbb{Z}}\sum_{s'=0}^{u-1}\sum_{b=0}^{u-1} x^{ps'}y^{ps'}q^{p(um-b)^2+p(um-b)s'}x^{2p(um-b)}\frac{x^{2pb}y^{2pb}q^{2pumb-2pb^2}-1}{1-x^uy^uq^{u(um-b)}}$$
$$=\sum_{m\in\mathbb{Z}}\sum_{s'=0}^{u-1}\sum_{b=0}^{u-1} x^{ps'}y^{ps'+2pb}q^{-pb^2-pbs'}\frac{q^{pu^2m^2+s'ump}x^{2pum}}{1-x^u(y^uq^{-bu})q^{mu^2}}$$
$$-\sum_{m\in\mathbb{Z}}\sum_{b=0}^{u-1} q^{p(um-b)^2}x^{2p(um-b)}\cdot\frac{x^{2pb}y^{2pb}q^{2pumb-2pb^2}-1}{1-x^uy^uq^{u(um-b)}}\cdot\sum_{s'=0}^{u-1}x^{ps'}y^{ps'}q^{p(um-b)s'}$$
$$=\sum_{s'=0}^{u-1}\sum_{b=0}^{u-1} x^{ps'}y^{ps'+2pb}q^{-pb^2-pbs'}\mathcal{K}_{2p}(q^{u^2},x^uq^{\frac{s'u}{2}},y^uq^{-\frac{s'u}{2}-bu})$$
$$+\sum_{m\in\mathbb{Z}}\sum_{b=1}^{u-1} q^{p(um-b)^2}x^{2p(um-b)}\cdot\frac{1-x^{2pb}y^{2pb}q^{2pumb-2pb^2}}{1-x^uy^uq^{u(um-b)}}\cdot\frac{1-x^{pu}y^{pu}q^{pu(um-b)}}{1-x^py^pq^{p(um-b)}} \ .$$

Note that the last double summation vanishes for $b=0$, so that we can omit the term $b=0$ in that summation. Now by swapping the denominators of both fractions inside the last term above and using(2.4.14) as,

$$\frac{1-[x^uy^uq^{u(um-b)}]^p}{1-x^uy^uq^{u(um-b)}} = \sum_{a=0}^{p-1}[x^uy^uq^{u(um-b)}]^a \ ,$$

and

$$\frac{1-[x^p y^p q^{p(um-b)}]^{2b}}{1-x^p y^p q^{p(um-b)}} = \sum_{c=0}^{2b-1} [x^p y^p q^{p(um-b)}]^c ,$$

(2.4.19)becomes,

$$\sum_{b=1}^{u-1}\sum_{a=0}^{p-1}\sum_{c=0}^{2b-1} q^{pb^2-aub-pbc} x^{au+cp} y^{au+cp} \sum_{m\in\mathbb{Z}} q^{pum^2-2pumb+amu^2+cpum} x^{2pum-2pbm} .$$

If we then change $c \to 2b-c$ and use (2.2.3) in the formula above, and finally take whatever we left aside during the calculation into account, we arrive at the following **period increasing formula for $\mathcal{K}_{(2p)}$**:

$$\begin{aligned}\mathcal{K}_{(2p)}(q,x,y) = \sum_{s'=0}^{u-1}\sum_{b=0}^{u-1} x^{ps'} y^{ps'+2pb} q^{-pb^2-pbs'} \mathcal{K}_{(2p)}(q^{u^2}, x^u q^{\frac{s'u}{2}}, y^u q^{-\frac{s'u}{2}-bu}) \\ + \sum_{\substack{r=1 \\ ur-ps>0}}^{p-1}\sum_{s=1}^{u-1} x^{ur-ps} y^{ur-ps} \theta(q^{2p}, x^{2p} q^{ur-ps}) \\ + \sum_{b=1}^{u-1}\sum_{a=0}^{p-1}\sum_{c=1}^{2b} x^{ua-pc} y^{2pb+ua-pc} q^{-pb^2-abu+pbc} \theta(q^{2pu^2}, x^{2pu} q^{au^2-puc}). \quad (2.4.20)\end{aligned}$$

The above relation is crucial in proving an extremely powerful identity, which plays a major role in the derivation of the S modular transformations of a large class of characters. A long and rather tedious road, sketched in Appendix A.3, leads to the following rewriting of the difference of higher-level Appell functions:

$$\boxed{\begin{aligned}&\mathcal{K}_{(2p)}(q,x,y) - \mathcal{K}_{(2p)}(q,x^{-1},y) = \\ &= \sum_{s=0}^{u-1}\sum_{b=1}^{u} x^{ps} y^{ps+2pb} q^{-pb^2-pbs} \\ &\times [\mathcal{K}_{(2p)}(q^{u^2}, x^u q^{\frac{su}{2}}, y^u q^{-\frac{su}{2}-bu}) - \mathcal{K}_{(2p)}(q^{u^2}, x^{-u} q^{-\frac{su}{2}}, y^u q^{-\frac{su}{2}-bu})]+ \\ &+ \sum_{\substack{b=1-u \\ 2pb+ps-ur>0}}^{u}\sum_{r=1}^{2p-1}\sum_{s=0}^{u-1} x^{ps-ur} y^{2pb+ps-ur} q^{-pb^2-pbs+bur} \Lambda_{(r,s+1,u,p)}(q^u, x^{-2u}) . \quad (2.4.21)\end{aligned}}$$

Where we have defined the following essential function,

$$\boxed{\Lambda_{(r,s,u,p)}(q,x) = \theta(q^{2pu}, x^p q^{ur-p(s-1)}) - q^{r(s-1)} x^{-r} \theta(q^{2pu}, x^p q^{-ur-p(s-1)}). \qquad (2.4.22)}$$

for which we can find some more discussions in appendix C.2.

2.4.4 More technical relations to theta functions

In what follows we show how some more technical relations and special combinations of higher-level Appell functions can be expressed through theta functions. We first note an identity showing that in a way -modulo rational expressions in theta functions- there is only one Appell function $\mathcal{K}_{(1)}$, with all the higher-level ones expressible through it,

$$\theta(q^p, xy^p)\mathcal{K}_{(p)}(q,z,y) - \sum_{r=0}^{p-1} z^r y^r \theta(q^p, z^p q^r)\mathcal{K}_{(1)}(q^p, x^{-1}y^{-p}, y^p q^{-r}) =$$
$$= -\theta(q^p, z^p x^{-1}) \frac{\vartheta_{(1,1)}(q, zyx)\, q^{-\frac{1}{8}} \tilde{\eta}(q)^3}{\vartheta_{(1,1)}(q, zy)\vartheta_{(1,1)}(q, x)}. \qquad (2.4.23)$$

This can be proved, e.g. by first noting that by the above open quasiperiodicity formulas, the left-hand side is in fact quasiperiodic in y (and obviously, in the other variables), and is therefore expressible as a ratio of theta functions. The actual theta functions in this ratio are found by matching the quasiperiodicity factors, and then the remaining q-dependent factor is fixed by comparing the residues of both sides. For an even level $2p$, it also follows that,

$$\sum_{b=0}^{p-1} x^{2b} q^{\frac{b^2}{p}} \left(\mathcal{K}_{(2p)}(q, xq^{\frac{b}{p}}, y) - \mathcal{K}_{(2p)}(q, x^{-1}q^{-\frac{b}{p}}, y)\right) =$$
$$= -\frac{\vartheta_{(1,1)}(q^{\frac{1}{p}}, x^2)\, q^{-\frac{1}{8p}}\, \tilde{\eta}(q^{\frac{1}{p}})^3}{\vartheta_{(1,1)}(q^{\frac{1}{p}}, xy)\vartheta_{(1,1)}(q^{\frac{1}{p}}, xy^{-1})}. \qquad (2.4.24)$$

To prove this, we use the same strategy as above, the crucial point being quasiperiodicity, which is shown as follows. With $\Delta_p f(q,x,y)$ temporarily denoting $f(q,x,yq) -$

$q^{\frac{p}{2}}y^p f(q,x,y)$, it follows from Eq.(2.4.10) that

$$\begin{aligned}
\Delta_p\big(\mathcal{K}_{(2p)}(q,xq^{\frac{b}{p}},y) - \mathcal{K}_{(2p)}(q,x^{-1}q^{-\frac{b}{p}},y)\big) = \\
=\sum_{a=1}^{p-1} x^{-a}q^{-\frac{ab}{p}+a}y^a(y^{2p-2a}q^{p-a}-1)\theta(q^{2p},x^{2p}q^{2b-a}) \\
+\sum_{a=1}^{p-1} x^{a}q^{\frac{ab}{p}+a}y^a(1-y^{2p-2a}q^{p-a})\theta(q^{2p},x^{2p}q^{2b+a}). \quad (2.4.25)
\end{aligned}$$

This also shows that $x^{2b}q^{\frac{b^2}{p}}\Delta_p\mathcal{K}_{2p}(q,x^{-1}q^{-\frac{b}{p}},y)$ depends on b only modulo p. In applying $\sum_{b=0}^{p-1} x^{2b}q^{b^2/p}$ to the second term in the right-hand side of (2.4.25), we can therefore make the shift $b \mapsto b-a$ without changing the summation limits for b. This readily implies that the left-hand side of (2.4.24) is quasiperiodic in y. And finally we introduce the last and certainly the most prominent relation between higher-level Appell functions and theta functions as,

$$\boxed{\sum_{m\in\mathbb{Z}} \mathcal{K}_{(p)}(q,z,yq^m)x^m = -\theta(q^p,z^px^{-1})\,\frac{\vartheta_{(1,1)}(q,zyx)\prod_{i\geq 1}(1-q^i)^3}{\vartheta_{(1,1)}(q,zy)\vartheta_{(1,1)}(q,x)}\,, \qquad (2.4.26)}$$

which is valid for $|q| < |x| < 1$. Appendix A.4 shows, how this crucial formula can be proven with the help of [11].

Chapter 3

Modular transformations of higher-level Appell functions

3.1 Introduction

Having obtained essential properties and relations among higher-level Appell functions in the last chapter, we now aim to determine their highly important modular transformation properties. A useful tool in doing so is to study their integrals over the cycles of a torus. This is the object of the first section and enables us to subsequently obtain an integral representation for the higher-level Appell functions $\mathcal{K}_p$. This representation is then used to calculate the sought after modular transformations. We will see that under the S modular transformation, the Appell functions yield an important function of two complex variables which we call $\Phi(\tau, \mu)$ and which will be thoroughly analysed in the last section.

3.2 Integration over tori cycles

We first recall how such integrations work out for the theta function $\vartheta(\tau, \lambda)$. Consider the two homotopically inequivalent $\boldsymbol{a}$-cycle and $\boldsymbol{b}$-cycle in Fig. (3.1) as the fundamental generators of a torus. It is well-known that the S transformation of the torus modulus τ interchanges these cycles [15] . So starting with the integration of $\vartheta(\tau, \lambda)$ along the $\boldsymbol{a}$-cycle namely,

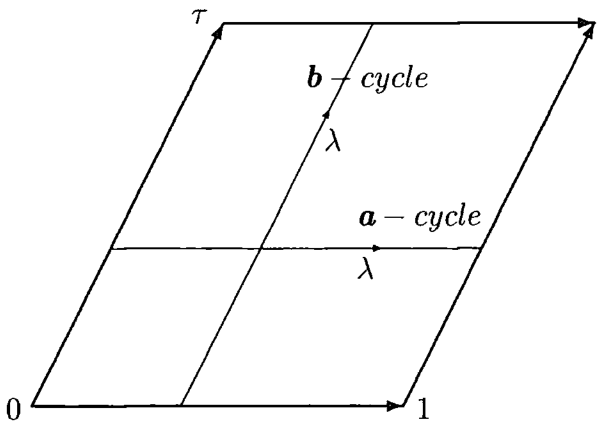

Figure 3.1: Integration along two fundamental cycles of a torus. They can be swapped by S-duality operator.

$$\oint_{\boldsymbol{a}} d\lambda\, \vartheta(\tau,\lambda) \stackrel{\text{def}}{=} \int_0^1 d\lambda\, \vartheta(\tau,\lambda) = 1 \ , \tag{3.2.1}$$

we can evaluate its S-dual integration along the $\boldsymbol{b}$-cycle of the torus as,

$$\oint_{\boldsymbol{b}} d\lambda\, \sqrt{-i\tau}\, e^{i\pi\frac{\lambda^2}{\tau}}\, \vartheta(\tau,\lambda) = \tau \ . \tag{3.2.2}$$

The following 'S-dualities' have been performed on (3.2.1), (recall (2.2.20))

$$\boldsymbol{a} - cycle \overset{S\text{–duality}}{\longrightarrow} \boldsymbol{b} - cycle$$

$$\vartheta(\tau,\lambda) \overset{S\text{–transformation}}{\longrightarrow} \sqrt{-i\tau}\, e^{i\pi\frac{\lambda^2}{\tau}}\vartheta(\tau,\lambda)$$

$$\mathit{1} \overset{S\text{–duality}}{\longrightarrow} \tau \ .$$

However it is still beneficial to check (3.2.2) by direct integration. Write

$$\oint_{\boldsymbol{b}} d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}\, \vartheta(\tau,\lambda) = \int_0^\tau d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}\, \vartheta(\tau,\lambda) = \sum_{m\in\mathbb{Z}} \int_{m\tau}^{(m+1)\tau} d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}, \tag{3.2.3}$$

where we have shifted the integration variable as $\lambda \to \lambda - m\tau$ in each term of the ϑ-series. For $\Im\tau > 0$, the integrals are defined by analytic continuation from $\tau = it$ with $t \in \mathbb{R}_{>0}$, and therefore,

$$\oint_{\boldsymbol{b}} d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}\, \vartheta(\tau,\lambda) = i\int_{\mathbb{R}} dx\, e^{-\pi\frac{x^2}{t}}\ \Big|_{t=-i\tau} = i\sqrt{-i\tau}\ , \tag{3.2.4}$$

which readily confirms what we got in (3.2.2) by using S-duality arguments.

Remarkably, much similarity is preserved if the theta functions are replaced with Appell functions in the above integrals. We concentrate on the **a** and **b**-cycle integrals of the level 1 Appell function $\mathcal{K}_1(\tau, \lambda - \mu, \mu)$ as the latter naturally emerges in the next section, where we discuss the modular transformations of higher-level Appell functions. First consider the following analog of (3.2.1),

$$\oint_{a} d\lambda\, \mathcal{K}_1(\tau, \lambda - \mu, \mu)) \stackrel{\text{def}}{=} \int_0^1 d\lambda\, \mathcal{K}_1(\tau, \lambda + i\varepsilon - \mu, \mu) = 1 \ , \qquad (3.2.5)$$

where the integral can be carried out as a contour integral in the complex λ plane and $\varepsilon > 0$ indicates how to bypass the singularities at $\lambda = 0$ and $\lambda = 1$.

Now consider

$$\oint_{b} d\lambda\, e^{i\pi\frac{\lambda^2 - 2\lambda\mu}{\tau}} \mathcal{K}_1(\tau, \lambda - \mu, \mu) \ . \qquad (3.2.6)$$

This is very much like $\oint_b d\lambda\, e^{i\pi\frac{\lambda^2}{\tau}}\, \vartheta(\tau, \lambda)$, and in the same way as the factor $e^{i\pi\frac{\lambda^2}{\tau}}$ is inherited from the behaviour of $\vartheta(\tau, \lambda)$ under the S modular transformation, the factor $e^{i\pi\frac{\lambda^2 - 2\lambda\mu}{\tau}}$ is also inherited from the behaviour of $\mathcal{K}_1(\tau, \lambda - \mu, \mu)$ under S, as we will see in Section 3.3. We have,

$$\oint_{b} d\lambda\, e^{i\pi\frac{\lambda^2 - 2\lambda\mu}{\tau}} \mathcal{K}_1(\tau, \lambda - \mu, \mu) \stackrel{\text{def}}{=} \int_0^\tau d\lambda\, e^{i\pi\frac{\lambda^2 - 2\lambda\mu}{\tau}}\, \mathcal{K}_1(\tau, \lambda + \varepsilon - \mu, \mu), \qquad (3.2.7)$$

where an infinitesimal positive real ε specifies the prescription to bypass the singularities. Again continuing from $\tau = it$ and $\mu = iy$ with positive real t and real y, we have, with $\lambda = ix$ and using (2.3.5),

$$\int_0^\tau d\lambda\, e^{i\pi\frac{\lambda^2 - 2\lambda\mu}{\tau}} \mathcal{K}_1(\tau, \lambda + \varepsilon - \mu, \mu) = i \sum_{m\in\mathbb{Z}} \int_0^t dx e^{-\pi\frac{x^2 - 2xy}{t}}\, \frac{e^{-\pi t m^2 - 2\pi m(x-y)}}{1 - e^{-2\pi(x+mt)} - i\varepsilon}\Bigg|_{\substack{t=-i\tau\\ y=-i\mu}} .$$

Making the same substitution $\lambda \to \lambda - m\tau$ as above, or $x \to x - mt$, and using the following identity,

$$\int_{-\infty}^{+\infty} dx\, f(x) = \fint_{-\infty}^{+\infty} dx\, f(x) + i\pi \operatorname*{res}_{x=0} f(x) \ ,$$

where $\fint_{-\infty}^{+\infty} dx\, f(x)$ is the principal value integral. One obtains with $f(x) = \frac{e^{-\frac{\pi}{t}x^2 + \frac{2\pi}{t}xy}}{1 - e^{-2\pi x}}$,

$$\oint_{b} d\lambda\, e^{i\pi\frac{\lambda^2 - 2\lambda\mu}{\tau}} \mathcal{K}_1(\tau, \lambda - \mu, \mu) = -i\sqrt{-i\tau}\, \Phi(\tau, \mu) \ , \qquad (3.2.8)$$

where,

$$\Phi(\tau,\mu) = -\frac{i}{2\sqrt{-i\tau}} - \frac{1}{2}\int_{\mathbb{R}} dx\, e^{-\pi x^2}\, \frac{\sinh\big(\pi x\sqrt{-i\tau}(1+2\frac{\mu}{\tau})\big)}{\sinh\big(\pi x\sqrt{-i\tau}\big)} \ . \tag{3.2.9}$$

Derivation shows that the same result is valid for the "$\boldsymbol{b}$"-integral (3.2.8) with a translated contour (which is homotopically unchanged) as,

$$\int_{\alpha\tau}^{\tau+\alpha\tau} d\lambda\, e^{i\pi\frac{\lambda^2-2\lambda\mu}{\tau}}\, \mathcal{K}_1(\tau,\lambda+\varepsilon-\mu,\mu) = -i\sqrt{-i\tau}\,\Phi(\tau,\mu), \qquad \alpha\in\mathbb{R}\ . \tag{3.2.10}$$

The above result will be very useful in the derivation of the S modular transformation of the higher-level Appell functions, as we now show.

3.3 Modular transformations of $\mathcal{K}_p$

If Appell functions could be rationally expressed through theta functions we would easily obtain their modular transformation by using (2.2.18)-(2.2.24). But based on what we realised in the last chapter, Appell functions satisfy open quasiperiodicity properties whereas theta functions are only quasiperiodic, and therefore, expressing Appell functions in terms of ratios of theta functions in a simple way is impossible. However, in view of the formula (2.4.26), one could represent Appell functions in terms of theta functions via an integral representation. This is precisely how we tackle the study of S transforms of higher-level Appell functions. Before going into the details of our derivation, we present our results, which are central to this work.

Under the action of the generators of modular transformations of the torus ($T: \tau\to\tau+1$, $S:\tau\to -1/\tau$), the higher-level Appell functions $\mathcal{K}_P(\tau,\nu,\mu)$ transform respectively as,

$$\mathcal{K}_p(\tau+1,\nu,\mu) = \begin{cases} \mathcal{K}_p(\tau,\nu\pm\frac{1}{2},\mu\mp\frac{1}{2}), & p \text{ odd}\ ,\\ \mathcal{K}_p(\tau,\nu,\mu), & p \text{ even.}\end{cases} \tag{3.3.1}$$

And

$$\begin{aligned}\mathcal{K}_p(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) &= \tau e^{i\pi p\frac{\nu^2-\mu^2}{\tau}}\,\mathcal{K}_p(\tau,\nu,\mu)\\ &\qquad + \tau\sum_{a=0}^{p-1} e^{i\pi p\frac{(\nu+\frac{a}{p}\tau)^2}{\tau}}\,\Phi(p\tau,p\mu-a\tau)\vartheta(p\tau,p\nu+a\tau)\ .\end{aligned} \tag{3.3.2}$$

As can be seen, the Φ function introduced in (3.2.9) appears as a transformation "kernel" in the second term. For the vast class of characters expressible in terms of higher-level Appell functions, a close study of the function Φ is necessary, in as much as one would like to understand the modular properties of these characters in some depth.

The behaviour under T is easily obtained from the original definition (2.3.5). We now proceed with the derivation of (3.3.2) in three main steps.

Step 1: Integral representation of $\mathcal{K}_p$

Recall formula (A.4.5),

$$\sum_{m\in\mathbb{Z}} \mathcal{K}_{(p)}(q,z,yq^m)x^m = -\theta(q^p,z^px^{-1})\,\frac{\vartheta_{(1,1)}(q,zyx)\,q^{-\frac{1}{8}}\tilde{\eta}(q)^3}{\vartheta_{(1,1)}(q,zy)\vartheta_{(1,1)}(q,x)}\,. \tag{3.3.3}$$

The right-hand side of this equation is a meromorphic function of x with poles at $x=q^n$, $n\in\mathbb{Z}$, but the identity holds in the annulus $\boldsymbol{A}_1=\{x\in\mathbb{C}\mid |q|<|x|<1\}$, where the left-hand side converges. We therefore temporarily assume that $x\in\boldsymbol{A}_1$ and then analytically continue the final result. Integrating over a closed contour C inside this annulus yields,

$$\mathcal{K}_{(p)}(q,z,y) = \frac{-1}{2i\pi}\oint_C \frac{dx}{x}\,\theta(q^p,z^px^{-1})\,\frac{\vartheta_{(1,1)}(q,zyx)\,q^{-\frac{1}{8}}\tilde{\eta}(q)^3}{\vartheta_{(1,1)}(q,zy)\vartheta_{(1,1)}(q,x)}. \tag{3.3.4}$$

We now rewrite this in the exponential notation $z=e^{2i\pi\nu}$, $y=e^{2i\pi\mu}$, $x=e^{2i\pi\lambda}$. The annulus $\boldsymbol{A}_1$ is then mapped into any of the parallelograms $\boldsymbol{P}_k$, $k\in\mathbb{Z}$, with the vertices $(k,k+1,k+1+\tau,k+\tau)$. We choose the rectangle P_0 in what follows. The integration contour C is then mapped into a contour in the interior of $\boldsymbol{P}_0$ connecting the points $\lambda=0$ and $\lambda=1$. Thus[1],

$$\mathcal{K}_p(\tau,\nu,\mu) = -e^{-\frac{i\pi}{4}\tau}\int_0^1 d\lambda\,\vartheta(p\tau,p\nu-\lambda)\,\frac{\vartheta_{1,1}(\tau,\nu+\mu+\lambda)\,\eta(\tau)^3}{\vartheta_{1,1}(\tau,\nu+\mu)\vartheta_{1,1}(\tau,\lambda+i\varepsilon)}\,, \tag{3.3.5}$$

where the infinitesimal real $\varepsilon>0$ gives the possibility of bypassing the singularities. This integral representation allows us to find the S transform of the higher-level

[1] Where λ now means $Re(\lambda)$.

Appell functions.

Step 2: S-transform of $\mathcal{K}_p$:

For this, we use the known S-transformation properties of the η and ϑ functions entering (3.3.5) with the result,

$$\mathcal{K}_p(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) = \sqrt{\tfrac{-i\tau}{p}}\,\tau\int_0^1 d\lambda\, e^{i\pi(\frac{p\nu^2}{\tau}+\frac{\lambda^2\tau}{p}+2\lambda\mu)}$$
$$\times -\sum_{r=0}^{p-1} e^{2i\pi r(\nu-\frac{\tau}{p}\lambda)+i\pi\frac{r^2}{p}\tau}\vartheta(p\tau,p\nu-\tau\lambda+r\tau)\frac{\vartheta_{1,1}(\tau,\nu+\mu+\tau\lambda)\,e^{-\frac{i\pi\tau}{4}}\eta(\tau)^3}{\vartheta_{1,1}(\tau,\nu+\mu)\vartheta_{1,1}(\tau,\tau\lambda-\varepsilon)}, \quad (3.3.6)$$

in which we have also used (2.4.13) to write $\vartheta(\frac{\tau}{p},\nu-\lambda\frac{\tau}{p})$ as,

$$\vartheta(\frac{\tau}{p},\nu-\lambda\frac{\tau}{p}) = \sum_{r=0}^{p-1} e^{2i\pi r(\nu-\frac{\tau}{p}\lambda)+i\pi\frac{r^2}{p}\tau}\vartheta(p\tau,p\nu-\tau\lambda+r\tau).$$

Now using the quasiperiodicity relation at (2.2.13) to modify $\vartheta_{1,1}(\tau,\nu+\mu+\tau\lambda)$ and $\vartheta_{1,1}(\tau,\tau\lambda-\varepsilon)$ in (3.3.6) we obtain,

$$\begin{aligned}\vartheta_{1,1}(\tau,\nu+\mu+\tau\lambda) &= \vartheta_{1,1}(\tau,\nu+\mu+\tau(\lambda-r)+r\tau)\\ &= (-1)^r e^{-i\pi(r^2+r)\tau}e^{-2i\pi[\nu+\mu+\tau(\lambda-r)]r}\vartheta_{1,1}(\tau,\nu+\mu+\tau(\lambda-r))\end{aligned}$$

and

$$\begin{aligned}\vartheta_{1,1}(\tau,\lambda\tau-\varepsilon) &= \vartheta_{1,1}(\tau,(\lambda-r)\tau-\varepsilon+r\tau)\\ &= (-1)^r e^{-i\pi(r^2+r)\tau}e^{-2i\pi r(\lambda-r)\tau}\vartheta_{1,1}(\tau,(\lambda-r)\tau-\varepsilon).\end{aligned}$$

Hence,

$$\frac{\vartheta_{1,1}(\tau,\nu+\mu+\tau\lambda)\,e^{-\frac{i\pi\tau}{4}}\eta(\tau)^3}{\vartheta_{1,1}(\tau,\nu+\mu)\vartheta_{1,1}(\tau,\tau\lambda-\varepsilon)} = e^{-2i\pi(\nu+\mu)r}\frac{\vartheta_{1,1}(\tau,\nu+\mu+(\lambda-r)\tau)\,e^{-\frac{i\pi\tau}{4}}\eta(\tau)^3}{\vartheta_{1,1}(\tau,\nu+\mu)\vartheta_{1,1}(\tau,(\lambda-r)\tau-\varepsilon)}.$$

Therefore by putting the above equality in the equation (3.3.6) and then using the identity (2.4.23), we can write the second line of (3.3.6) as,

$$\sum_{r=0}^{p-1} e^{2i\pi(\nu-\frac{\tau}{p}\lambda)r+i\pi\frac{r^2}{p}\tau-2i\pi(\nu+\mu)r}\Big[\vartheta(p\tau,(\lambda-r)\tau+p\mu)\mathcal{K}_p(\tau,\nu,\mu)-$$
$$\sum_{a=0}^{p-1} e^{2i\pi a(\nu+\mu)}\vartheta(p\tau,p\nu+a\tau)\mathcal{K}_1(p\tau,(r-\lambda)\tau+\varepsilon-p\mu,p\mu-a\tau)\Big]. \quad (3.3.7)$$

Now inserting (3.3.7) into (3.3.6) results in having two different terms, say $\mathcal{T}_1$ and $\mathcal{T}_2$. The first one can be simplified to,

$$\mathcal{T}_1 = \sqrt{\tfrac{-i\tau}{p}}\,\tau \int_0^1 d\lambda\, e^{i\pi(\frac{p\nu^2}{\tau}+\frac{\lambda^2\tau}{p}+2\lambda\mu)}\,\vartheta(\frac{\tau}{p},\mu+\lambda\frac{\tau}{p})\,\mathcal{K}_p(\tau,\nu,\mu)\ , \tag{3.3.8}$$

in which by using (2.4.13), the following simplification has been also done,

$$\sum_{r=0}^{p-1}\vartheta(p^2(\frac{\tau}{p}),-p(\mu+\lambda\frac{\tau}{p})+rp(\frac{\tau}{p}))e^{-2i\pi(\mu+\lambda\frac{\tau}{p})r+i\pi r^2\frac{\tau}{p}} = \vartheta(\frac{\tau}{p},\mu+\lambda\frac{\tau}{p}). \tag{3.3.9}$$

Changing the variable $\lambda \to \lambda\frac{p}{\tau}$ in (3.3.8) gives,

$$\mathcal{T}_1 = \sqrt{-ip\tau}\, e^{i\pi p\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_p(\tau,\nu,\mu)\int_0^{\frac{\tau}{p}} d\lambda e^{i\pi\frac{(\lambda+\mu)^2}{\tau/p}}\,\vartheta(\frac{\tau}{p},\lambda+\mu)\ . \tag{3.3.10}$$

The second term, $\mathcal{T}_2$,is given by,

$$\begin{aligned}\mathcal{T}_2 = -\sqrt{\tfrac{-i\tau}{p}}\tau &\int_0^1 d\lambda\, e^{i\pi(\frac{p\nu^2}{\tau}+\frac{\lambda^2\tau}{p}+2\lambda\mu)}\sum_{r=0}^{p-1}e^{i\pi\frac{r^2}{p}\tau-2i\pi r(\mu+\lambda\frac{\tau}{p})}\\ &\times\sum_{a=0}^{p-1}e^{2i\pi a(\nu+\mu)}\vartheta(p\tau,p\nu+a\tau)\mathcal{K}_1(p\tau,(r-\lambda)\tau+\varepsilon-p\mu,p\mu-a\tau)\ .\end{aligned} \tag{3.3.11}$$

It can be rewritten using the change of variable $\lambda \to \frac{\lambda}{\tau}+r$ in what follows,

$$\begin{aligned}\mathcal{T}_2 = -\sqrt{\tfrac{-i\tau}{p}}e^{i\pi\frac{p\nu^2}{\tau}}&\sum_{a=0}^{p-1}e^{2i\pi a(\nu+\mu)}\vartheta(p\tau,p\nu+a\tau)\\ &\times\sum_{r=0}^{p-1}\int_{-r\tau}^{-r\tau+\tau} d\lambda e^{i\pi\frac{\lambda^2}{p\tau}+2i\pi\frac{\lambda\mu}{\tau}}\,\mathcal{K}_1(p\tau,-\lambda+i\varepsilon-p\mu,p\mu-a\tau).\end{aligned}$$

We do summations and get,

$$\begin{aligned}\mathcal{T}_2 = -\sqrt{\tfrac{-i\tau}{p}}e^{i\pi\frac{p\nu^2}{\tau}}&\sum_{a=0}^{p-1}e^{2i\pi a(\nu+\mu)}\vartheta(p\tau,p\nu+a\tau)\\ &\times\int_{-(p-1)\tau}^{\tau} d\lambda e^{i\pi\frac{\lambda^2}{p\tau}+2i\pi\frac{\lambda\mu}{\tau}}\,\mathcal{K}_1(p\tau,-\lambda+i\varepsilon-p\mu,p\mu-a\tau)\ .\end{aligned}$$

The change of variable $\lambda \to -(\lambda+a\tau)$ finally gives us,

$$\begin{aligned}\mathcal{T}_2 = -\sqrt{\tfrac{-i\tau}{p}}&\sum_{a=0}^{p-1} e^{i\pi p\frac{(\nu+\frac{a}{p}\tau)^2}{\tau}}\vartheta(p\tau,p\nu+a\tau)\\ \times&\int_{-\tau(1+a)}^{-\tau(1+a-p)} d\lambda e^{i\pi\frac{\lambda^2}{p\tau}-2i\pi\frac{\lambda(p\mu-a\tau)}{p\tau}}\,\mathcal{K}_1(p\tau,\lambda-i\varepsilon-(p\mu-a\tau),p\mu-a\tau).\end{aligned} \tag{3.3.12}$$

Step 3: Performing the integrations in $\mathcal{T}_1$ and $\mathcal{T}_2$

We first want to show how using formula (3.2.10) can help us compute the integration part of (3.3.12). However before that we need to slightly change (3.2.10) to be clearly comparable with our choice. To do that, we change $\tau \to p\tau$ and $\mu \to p\mu - a\tau$ in (3.2.10) and obtain,

$$\int_{\alpha p\tau}^{p\tau+\alpha p\tau} d\lambda\, e^{i\pi\frac{\lambda^2-2\lambda(p\mu-a\tau)}{p\tau}}\, \mathcal{K}_1(p\tau, \lambda+\varepsilon-(p\mu-a\tau), p\mu-a\tau) = -i\sqrt{-ip\tau}\, \Phi(p\tau, p\mu-a\tau). \tag{3.3.13}$$

Therefore just by choosing $\alpha p = -(1+a)$ we will be exactly on (3.3.12). Thus by replacing the whole integral in (3.3.12) with the right hand side of (3.3.13) we arrive at,

$$\mathcal{T}_2 = \tau \sum_{a=0}^{p-1} e^{i\pi p\frac{(\nu+\frac{a}{p}\tau)^2}{\tau}}\, \Phi(p\tau, p\mu - a\tau)\, \vartheta(p\tau, p\nu + a\tau)\ . \tag{3.3.14}$$

Furthermore, the integration in (3.3.10) can be similarly derived, using the argument presented in (3.2.4) while studying theta functions on a torus. In fact by changing $\tau \to \frac{\tau}{p}$ and $\lambda \to \lambda + \mu$ in (3.2.4), $\mathcal{T}_1$ in (3.3.10) will be readily computed as,

$$\begin{aligned} \mathcal{T}_1 &= \sqrt{-ip\tau}\, e^{i\pi p\frac{\nu^2-\mu^2}{\tau}} \mathcal{K}_p(\tau, \nu, \mu) \cdot \left(- i\sqrt{-i\tfrac{\tau}{p}}\right) \\ &= \tau\, e^{i\pi p\frac{\nu^2-\mu^2}{\tau}} \mathcal{K}_p(\tau, \nu, \mu). \end{aligned} \tag{3.3.15}$$

Finally, adding the two terms $\mathcal{T}_1$ and $\mathcal{T}_2$ gives the expression (3.3.2) for the S transform of the higher-level Appell functions $\mathcal{K}_p$ in terms of $\mathcal{K}_p$ itself and a linear combination involving shifts of the theta function $\vartheta(p\tau, p\nu)$.

In the light of the general discussion offered in the Introduction, culminating with formula (1.35), we see how one may construct the matrix automorphy factor for the transformation $\mathcal{S} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The key expressions are (3.3.2), which we rewrite in terms of the components of the vector $\vartheta^{(\mathbf{p})}(\tau,\nu)$ (see (1.31)),

$$\mathcal{K}_p(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) = \tau e^{i\pi p\frac{\nu^2-\mu^2}{\tau}}\,\mathcal{K}_p(\tau,\nu,\mu) + \tau\sum_{a=0}^{p-1} e^{i\pi p\frac{\nu^2}{\tau}}\,\Phi(p\tau,p\mu-a\tau)\vartheta_a^{(p)}(\tau,\nu)\ , \tag{3.3.16}$$

as well as,

$$\vartheta_r^{(p)}(-\frac{1}{\tau},\frac{\nu}{\tau}) = \sqrt{\frac{-i\tau}{p}}e^{i\pi\frac{\nu^2}{\tau}}\sum_{r'=0}^{p-1} e^{-2i\pi\frac{rr'}{p}}\vartheta_{r'}^{(p)}(\tau,\nu)\ . \tag{3.3.17}$$

The latter transformation formula is obtained as follows. Starting from the definition (1.31) and performing an S-transformation we have,

$$\begin{aligned}
\vartheta_r^{(p)}(-\frac{1}{\tau},\frac{\nu}{\tau}) &= e^{2i\pi r\frac{\nu}{\tau}}e^{-i\pi\frac{r^2}{p\tau}}\vartheta(-\frac{p}{\tau},\frac{p\nu-r}{\tau}) \\
&\overset{using(2.2.20)}{=} \sqrt{-\frac{i\tau}{p}}e^{i\pi p\frac{\nu^2}{\tau}}\vartheta(\frac{\tau}{p},\nu-\frac{r}{p}) \\
&\overset{using(2.4.13)}{=} \sqrt{-\frac{i\tau}{p}}e^{i\pi p\frac{\nu^2}{\tau}}\sum_{r'=0}^{p-1} e^{2i\pi r'(\nu-\frac{r}{p})}e^{i\pi\tau\frac{r'^2}{p}}\vartheta(p\tau,p\nu-r+r'\tau) \\
&\overset{using(2.2.3)}{=} \sqrt{-\frac{i\tau}{p}}e^{i\pi p\frac{\nu^2}{\tau}}\sum_{r'=0}^{p-1} e^{2i\pi r'(\nu-\frac{r}{p})}e^{i\pi\tau\frac{r'^2}{p}}\vartheta(p\tau,p\nu+r'\tau) \\
&\overset{using(1.31)}{=} \sqrt{-\frac{i\tau}{p}}e^{i\pi p\frac{\nu^2}{\tau}}\sum_{r'=0}^{p-1} e^{-2i\pi\frac{rr'}{p}}\vartheta_{r'}^{(p)}(\tau,\nu)\ .
\end{aligned} \tag{3.3.18}$$

The information presented in the formulas (3.3.16) and (3.3.17) may be encoded in the following expression, which uses the definition (1.30) and introduces the matrix,

$$\mathbf{J}_p^{-1}(\mathbb{K}_p,\mathbb{S};\tau,\nu,\mu) = e^{i\pi p\frac{\nu^2}{\tau}}\begin{pmatrix} \tau e^{-i\pi p\frac{\mu^2}{\tau}} & \tau\mathbf{\Psi} \\ \mathbf{0} & (-i\tau)^{\frac{1}{2}}\mathbf{D}_p^*(\mathcal{S}), \end{pmatrix} \tag{3.3.19}$$

where $\mathbf{D_p}(\mathcal{S})$ has elements,

$$D_{mn} = \frac{1}{\sqrt{p}}e^{2i\pi\frac{nm}{p}}, \qquad n,m=0,1,\ldots p-1, \tag{3.3.20}$$

and $\mathbf{\Psi}$ is the row vector with components $(\mathbf{\Psi})_a = \Phi(p\tau,p\mu-a\tau), a=0,1,\ldots,p-1$. We therefore write,

$$\mathbb{K}_p(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau}) = \mathbf{J}_p^{-1}(\mathbb{K}_p,\mathcal{S};\tau,\nu,\mu)\mathbb{K}_p(\tau,\nu,\mu), \tag{3.3.21}$$

or again (recall (1.35)),

$$\mathcal{S}\cdot\mathbb{K}_p(\tau,\nu,\mu)=\mathbf{J}_p(\mathbb{K}_p,\mathcal{S};\tau,\nu,\mu)\mathbb{K}_p(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})=\mathbb{K}_p(\tau,\nu,\mu) \tag{3.3.22}$$

with the matrix automorphy factor for $\mathcal{S}$ being,

$$\mathbf{J}_p(\mathbb{K}_p,\mathcal{S};\tau,\nu,\mu)=e^{-i\pi p\frac{\nu^2}{\tau}}\begin{pmatrix}\tau^{-1}e^{i\pi p\frac{\mu^2}{\tau}} & -(-i\tau)^{-\frac{1}{2}}e^{i\pi p\frac{\mu^2}{\tau}}\mathbf{\Psi}\mathbf{D}_p(\mathbb{S})\\ \mathbf{0} & (-i\tau)^{-\frac{1}{2}}\mathbf{D}_p(\mathbb{S})\end{pmatrix}. \tag{3.3.23}$$

3.4 The Φ function and its properties

We now study important properties of the function Φ appearing in the S transform of the higher-level Appell functions. These properties are essential in understanding the modular behaviour of models whose partition functions might include non-quasiperiodic characters and therefore, higher-level Appell functions. As in the case of theta and Appell functions, we discuss basic and open quasiperiodicity properties of the function Φ, and introduce scaling laws (similar to period increasing statements in the last chapter). We end this section by giving the S-modular transformation formula for Φ.

We find it useful to introduce the associated function $\phi(\tau,\mu)$,

$$\phi(\tau,\mu)=\frac{i}{\sqrt{-i\tau}}\int_0^{\tau}d\lambda\, e^{i\pi\frac{\lambda^2-2\lambda\mu}{\tau}}\mathcal{K}_1(\tau,\lambda-\mu,\mu)\ , \tag{3.4.1}$$

or equivalently,

$$\phi(\tau,\mu)=\Phi(\tau,\mu)+\frac{i}{2\sqrt{-i\tau}}\ , \tag{3.4.2}$$

$$=-\frac{1}{2}\int_{\mathbb{R}}dx\, e^{-\pi x^2}\frac{\sinh\big(\pi x\sqrt{-i\tau}(1+2\frac{\mu}{\tau})\big)}{\sinh\big(\pi x\sqrt{-i\tau}\big)}\ , \tag{3.4.3}$$

as it gives more concise expressions for some formulas. It is worth noting that almost all the properties mentioned above can be derived from the integral representation (3.4.3), or alternatively using (3.3.2) with $p=1$, namely,

$$\mathcal{K}_1(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau})=\tau e^{i\pi\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_1(\tau,\nu,\mu)+\tau\, e^{i\pi\frac{\nu^2}{\tau}}\Phi(\tau,\mu)\vartheta(\tau,\nu). \tag{3.4.4}$$

Another valuable source of information for the study of the Φ function is the recent literature on Barnes-like special functions, arising in various problems (see [16], [17], [18] and also [19], [20], [21], [22]).

3.4.1 Basic and open quasiperiodic properties

First, the expression of $\phi(\tau,\mu)$ at $\mu = \frac{m\tau}{2}$, $m \in \mathbb{Z}$ is easily obtained from the initial definition (3.4.3). It yields,

$$\begin{aligned} \phi(\tau, \tfrac{m\tau}{2}) &= -\frac{1}{2}\sum_{j=0}^{m} e^{-i\pi\tau\frac{(m-2j)^2}{4}}, \quad m > 0, \\ \phi(\tau, -\tfrac{m\tau}{2}) &= \frac{1}{2}\sum_{j=1}^{m-1} e^{-i\pi\tau\frac{(m-2j)^2}{4}}, \quad m > 1, \end{aligned} \tag{3.4.5}$$

and $\phi(\tau,0) = -\frac{1}{2}$, $\phi(\tau,-\tau/2) = 0$.

Next, by elementary transformations, we find that ϕ satisfies the equations,

$$\begin{aligned} \phi(\tau,\mu+m\tau) &= \phi(\tau,\mu) - \sum_{j=1}^{m} e^{-i\pi\frac{(\mu+j\tau)^2}{\tau}}, \\ \phi(\tau,\mu-m\tau) &= \phi(\tau,\mu) + \sum_{j=0}^{m-1} e^{-i\pi\frac{(\mu-j\tau)^2}{\tau}}, \end{aligned} \qquad m \in \mathbb{N}\,. \tag{3.4.6}$$

And hence, the same relations are valid for Φ. These open quasiperiodicity relations are similar to those for $\mathcal{K}_p$. We call them "dual" open quasiperiodicity properties as they are changed by changing the sign of m. A slightly more involved calculation with the integral representation in (3.4.3), leads to other dual open quasiperiodic relations, which are more conveniently written in terms of Φ,

$$\begin{aligned} \Phi(\tau,\mu+m) &= e^{-i\pi\frac{m^2}{\tau}-2i\pi m\frac{\mu}{\tau}}\,\Phi(\tau,\mu) + \frac{i}{\sqrt{-i\tau}}\sum_{j=1}^{m} e^{i\pi\frac{j(j-2m)}{\tau}-2i\pi j\frac{\mu}{\tau}}, \\ \Phi(\tau,\mu-m) &= e^{-i\pi\frac{m^2}{\tau}+2i\pi m\frac{\mu}{\tau}}\,\Phi(\tau,\mu) - \frac{i}{\sqrt{-i\tau}}\sum_{j=0}^{m-1} e^{i\pi\frac{j(j-2m)}{\tau}+2i\pi j\frac{\mu}{\tau}}, \end{aligned} \tag{3.4.7}$$

where $m \in \mathbb{N}$. We explicitly show how to obtain the first of the above relations. Recall the analytic continuation prescription and consider $\Phi(it,\mu)$ with $t \in \mathbb{R}_+$. We write

$$\Phi(it,\mu) = -\int_{\mathbb{R}-i0} dx\, e^{-\pi x^2}\,\frac{e^{\pi x\sqrt{t}(1+2\frac{\mu}{it})}}{e^{\pi x\sqrt{t}} - e^{-\pi x\sqrt{t}}}\,, \tag{3.4.8}$$

and consider $\Phi(it,\mu+m)$ with $m \in \mathbb{N}$, changing the integration variable to $x' = x + i\frac{m}{\sqrt{t}}$. This gives,

$$\Phi(it,\mu+m) = e^{-\pi\frac{m^2}{t}-2\pi\frac{m\mu}{t}}\,\Phi_{\frac{im}{\sqrt{t}}}(it,\mu)\,,$$

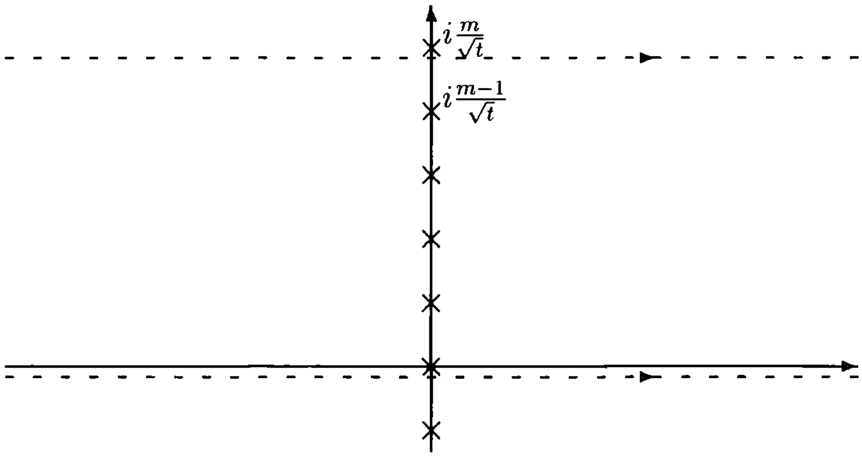

Figure 3.2: Integration contours for $\Phi(it,\mu)$ (the lower dashed line) and $\Phi_{\frac{im}{\sqrt{t}}}(it,\mu)$ (the upper dashed line) in the complex x plane and poles of the integrand (crosses).

where

$$\Phi_{\frac{im}{\sqrt{t}}}(it,\mu) = -\int\limits_{\mathbb{R}+i\frac{m}{\sqrt{t}}-i0} dx\, e^{-\pi x^2}\,\frac{e^{\pi x\sqrt{t}(1+2\frac{\mu}{it})}}{e^{\pi x\sqrt{t}}-e^{-\pi x\sqrt{t}}}.$$

A residue calculation in accordance with

$$\Phi_{\frac{im}{\sqrt{t}}}(it,\mu) = \Phi(it,\mu) - 2i\pi\sum_{n=0}^{m-1}\operatorname*{res}_{x=i\frac{n}{\sqrt{t}}}\left(-e^{-\pi x^2}\,\frac{e^{\pi x\sqrt{t}(1+2\frac{\mu}{it})}}{e^{\pi x\sqrt{t}}-e^{-\pi x\sqrt{t}}}\right)$$

(see Fig 3.2), then yields the first equation in (3.4.7). Alternatively, the functional equations (3.4.7) can be also deduced from (3.4.4) and the corresponding property of the Appell function in Eq. (2.4.3). Next, a "reflection property" follows from (2.4.2) (or equivalently, can be directly derived from (3.4.3)),

$$\Phi(\tau,-\mu) = \frac{-i}{\sqrt{-i\tau}} - e^{-i\pi\frac{\mu^2}{\tau}} - \Phi(\tau,\mu)\ . \tag{3.4.9}$$

It can be written in a slightly different form,

$$\Phi(\tau,-\mu-\tau) = -e^{2i\pi\frac{\mu+\frac{1}{2}}{\tau}}\,\Phi(\tau,\mu+1)\ . \tag{3.4.10}$$

3.4.2 The Φ function's scaling laws

We intend now to introduce some new properties of the Φ function. They cannot be precisely called period increasing properties, since they are not as genuinely periodic as functions we studied in the last chapter. We call them "scaling laws", although they are very similar to the period increasing formulas of Appell functions. Their derivation is quite involved, and we present some details below, with the help of an appendix. First use the relations (2.4.15) and (3.4.4) to write,

$$\Phi(\tau,\mu) = \sum_{b=0}^{u-1} \Phi(u^2\tau, u\mu - bu\tau). \tag{3.4.11}$$

Now by using the following identity, proved in Appendix B.1 when u and p are coprime positive integers,

$$\sum_{r=1}^{2p}\sum_{s=1}^{u} \frac{q^{pu-u(r-1)-p(s-1)}}{q^{pu}-q^{-pu}} = \frac{q}{q-1} - \sum_{\substack{r=1\\ ur-ps>0}}^{p-1}\sum_{s=1}^{u-1} q^{-ur+ps}, \tag{3.4.12}$$

the most important scaling law for Φ function reads,

$$\Phi(\tfrac{\tau}{2pu},\tfrac{\mu}{2pu}) = -\sum_{\substack{r=1\\ ur-ps>0}}^{p-1}\sum_{s=1}^{u-1} e^{-i\pi\frac{\tau}{2pu}(\frac{\mu}{\tau}-ur+ps)^2}$$
$$+\sum_{r=1}^{2p}\sum_{s=1}^{u}\Phi(2pu\tau,\mu-u(r-1)\tau-p(s-1)\tau). \tag{3.4.13}$$

However, in studying the S transform of $N=2$ and $\widehat{s\ell}(2|1)$ non-unitary characters, we will need a scaled version of (3.3.2), which involves the following scaling formula,

$$\sum_{a=0}^{u-1} e^{i\pi\frac{a^2\tau}{u}+2i\pi a\nu}\Phi(u\tau,u\mu-a\tau)\,\vartheta(u\tau,u\nu+a\tau) =$$
$$= \tfrac{1}{u}\sum_{a=0}^{u-1} e^{i\pi\frac{a^2}{u\tau}+2i\pi a\frac{\mu}{\tau}}\Phi(\tfrac{\tau}{u},\mu+\tfrac{a}{u})\,\vartheta(\tfrac{\tau}{u},\nu-\tfrac{a}{u}), \quad u\in\mathbb{N}\,. \tag{3.4.14}$$

The remaining of this subsection is dedicated to the proof of the above formula. First change $\tau\to\frac{\tau}{u}$ and $\mu\to\mu+\frac{a}{u}$ in (3.4.11) for any positive integer "a". This gives,

$$\Phi(\frac{\tau}{u},\mu+\frac{a}{u}) = \sum_{b=0}^{u-1}\Phi(u\tau,u\mu+a-b\tau)\,. \tag{3.4.15}$$

Meanwhile, under the change $\tau\to\frac{\tau}{u}$ and $\nu\to\nu-\frac{a}{u}$, the formula (2.4.13) becomes,

$$\vartheta(\frac{\tau}{u},\nu-\frac{a}{u}) = \sum_{j=0}^{u-1} e^{2i\pi(\nu-\frac{a}{u})j+i\pi\frac{\tau}{u}j^2}\vartheta(u\tau,u\nu+j\tau)\,. \tag{3.4.16}$$

Using (3.4.15) and (3.4.16) in the right hand side ($R.H.S.$) of (3.4.14) gives,

$$R.H.S = \frac{1}{u}\sum_{a=0}^{u-1} e^{i\pi\frac{a^2}{u\tau}+2i\pi\frac{a\mu}{\tau}} \sum_{b=0}^{u-1}\Phi(u\tau, u\mu - b\tau + a)$$
$$\times\sum_{c=0}^{u-1} e^{2i\pi(\nu-\frac{a}{u})c+i\pi\frac{\tau}{u}c^2}\vartheta(u\tau, u\nu + c\tau). \quad (3.4.17)$$

Also it appears from (3.4.7) that,

$$\Phi(u\tau, u\mu - b\tau + a) = e^{-i\pi\frac{a^2}{u\tau}-2i\pi a\frac{u\mu-b\tau}{u\tau}}\,\Phi(u\tau, u\mu - b\tau) + \frac{i}{\sqrt{-iu\tau}}\sum_{j=1}^{a} e^{i\pi\frac{j(j-2a)}{u\tau}-2i\pi j\frac{u\mu-b\tau}{u\tau}}\ , \quad (3.4.18)$$

where $a \in \mathbb{N}$. We do need now to insert this formula in (3.4.17), however (3.4.18) only holds for non-zero positive integer amounts of "a". We therefore write,

$$R.H.S = \frac{1}{u}\sum_{a=1}^{u-1}\sum_{b=0}^{u-1} e^{2i\pi a\frac{b}{u}}\Phi(u\tau, u\mu - b\tau)\sum_{c=0}^{u-1} e^{2i\pi(\nu-\frac{a}{u})c+i\pi\frac{\tau}{u}c^2}\vartheta(u\tau, u\nu + c\tau)$$
$$+ \frac{i}{\sqrt{-iu\tau}}\frac{1}{u}\sum_{a=0}^{u-1} e^{i\pi\frac{a^2}{u\tau}+2i\pi\frac{a\mu}{\tau}}\sum_{b=0}^{u-1}\sum_{j=1}^{a} e^{i\pi\frac{j(j-2a)}{u\tau}-2i\pi j\frac{u\mu-b\tau}{u\tau}}\sum_{c=0}^{u-1} e^{2i\pi(\nu-\frac{a}{u})c+i\pi\frac{\tau}{u}c^2}\vartheta(u\tau, u\nu + c\tau)$$
$$+ \frac{1}{u}\sum_{b=0}^{u-1} e^{2i\pi\nu c+i\pi\frac{\tau}{u}c^2}\Phi(u\tau, u\mu - b\tau)\vartheta(u\tau, u\nu + c\tau). \quad (3.4.19)$$

Where the last term is the $a = 0$ contribution. But in the second term above, the only computation for index "b" is $\sum_{b=0}^{u-1} e^{2i\pi j\frac{b}{u}}$ which is actually zero, since $1 \le j \le u-1$ (using lemma (2.4.14)). Therefore the second term in (3.4.19)vanishes. The first term in (3.4.19) can also be simplified to be,

$$\frac{1}{u}\sum_{b=0}^{u-1}\sum_{c=0}^{u-1} e^{2i\pi\nu c+i\pi\frac{\tau}{u}c^2}\Phi(u\tau, u\mu - b\tau)\vartheta(u\tau, u\nu + c\tau)\sum_{a=1}^{u-1} e^{2i\pi\frac{b-c}{u}a}\ , \quad (3.4.20)$$

and since $-(u-1) \le b - c \le u - 1$, the last summation in the expression above reads (using (2.4.14)),

$$\sum_{a=1}^{u-1} e^{2i\pi\frac{b-c}{u}a} = u\,\delta(b-c) - 1\ .$$

Taking this equality into account, the first term in (3.4.19) will finally be,

$$\sum_{c=0}^{u-1} e^{i\pi\frac{\tau}{u}a^2+2i\pi\nu c}\Phi(u\tau, u\mu - c\tau)\vartheta(u\tau, u\nu + c\tau)$$
$$- \frac{1}{u}\sum_{b=0}^{u-1} e^{2i\pi\nu c+i\pi\frac{\tau}{u}c^2}\Phi(u\tau, u\mu - b\tau)\vartheta(u\tau, u\nu + c\tau),$$

in which the second term clearly cancels with the third term in formula (3.4.19). Therefore we finally arrive at (3.4.14), which also allows us to rewrite the S modular transformation (3.3.2) of level p Appell functions as,

$$\boxed{\begin{aligned}\mathcal{K}_p(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) &= \tau\, e^{i\pi p\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_p(\tau,\nu,\mu)\\ &\quad+\tfrac{\tau}{p}\sum_{a=0}^{p-1} e^{i\pi\frac{p}{\tau}\nu^2+i\pi\frac{a^2}{p\tau}+2i\pi a\frac{\mu}{\tau}}\Phi(\tfrac{\tau}{p},\mu+\tfrac{a}{p})\,\vartheta(\tfrac{\tau}{p},\nu-\tfrac{a}{p}). \qquad (3.4.21)\end{aligned}}$$

This is the scaled version of (3.3.2) used in subsequent chapters.

3.4.3 S modular transformation of the Φ function

We finish this chapter by calculating the S-transformation of the Φ function, along with its verification in asymptotic ranges. We start by acting with another S transform on formula (3.4.4) and we use (2.2.20) to obtain,

$$\mathcal{K}_1(\tau,-\nu,-\mu) = \frac{-1}{\tau}e^{-i\pi\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_1(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) - \frac{\sqrt{-i\tau}}{\tau}\Phi(-\tfrac{1}{\tau},\tfrac{\mu}{\tau})\vartheta(\tau,\nu)\ .$$

We rewrite $\mathcal{K}_1(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})$ in the above formula with the help of (3.4.4) and get,

$$\mathcal{K}_1(\tau,-\nu,-\mu) = -\mathcal{K}_1(\tau,\nu,\mu) - e^{i\pi\frac{\mu^2}{\tau}}\Phi(\tau,\mu)\vartheta(\tau,\nu) - \frac{\sqrt{-i\tau}}{\tau}\Phi(-\tfrac{1}{\tau},\tfrac{\mu}{\tau})\vartheta(\tau,\nu)\ . \tag{3.4.22}$$

Now by just using the following elementary identity,

$$\mathcal{K}_1(\tau,\nu,\mu) = -\mathcal{K}_1(\tau,-\nu,-\mu)+\vartheta(\tau,\nu), \tag{3.4.23}$$

or alternatively, using the identities (2.4.2) and (2.2.7) to cancel the presence of Appell functions in (3.4.22) we arrive at,

$$\Phi(-\tfrac{1}{\tau},\tfrac{\mu}{\tau}) = -i\sqrt{-i\tau}\big(e^{i\pi\frac{\mu^2}{\tau}}\,\Phi(\tau,\mu)+1\big), \tag{3.4.24}$$

which, in view of (3.4.2),we can also rewrite as,

$$\phi(-\tfrac{1}{\tau},\tfrac{\mu}{\tau}) = -i\sqrt{-i\tau}\big(e^{i\pi\frac{\mu^2}{\tau}}\,\phi(\tau,\mu)+\frac{1}{2}\big)-\frac{1}{2}e^{i\pi\frac{\mu^2}{\tau}}. \tag{3.4.25}$$

It is instructive to verify this formula by comparing the asymptotic expansions of the integral in (3.4.3) as $\tau \to \infty$ and then as $-i\tau \searrow 0$. We first find the asymptotic expansion of $\phi(it, iy)$ for large positive t. We write,

$$\phi(it,iy) \overset{t\to+\infty}{\asymp} -\int_{\epsilon}^{+\infty} dx\, e^{-\pi x^2} \frac{\sinh\left(\pi x\sqrt{t}(1+2\frac{y}{t})\right)}{\sinh\left(\pi x\sqrt{t}\right)}$$

and use (2.4.14) with $A = e^{-\pi x\sqrt{t}}$, $|A| \leq 1$, to obtain

$$\phi(it,iy) \overset{t\to+\infty}{\asymp} -2\sum_{m=0}^{\infty}\int_{\epsilon}^{+\infty} dx\, e^{-\pi x^2} \sinh\Big(\pi x(\sqrt{t}+2\tfrac{y}{\sqrt{t}})\Big)\; e^{-\pi x(2m+1)\sqrt{t}}.$$

By expanding the sinh function we obtain,

$$\phi(it,iy) \overset{t\to+\infty}{\asymp} \sum_{m=0}^{\infty}\int_{\epsilon}^{+\infty} dx[e^{-\pi x^2-\pi x(\sqrt{t}+\frac{2y}{\sqrt{t}}+(2m+1)\sqrt{t})} - e^{-\pi x^2+\pi x(\sqrt{t}+\frac{2y}{\sqrt{t}}-(2m+1)\sqrt{t})}]. \tag{3.4.26}$$

The above integral can be evaluated in terms of the complementary error function defined as,

$$\mathrm{erfc}(z) = 1-\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} dx\, e^{-x^2}.$$

Indeed, note that

$$\int_{\epsilon}^{+\infty} dx e^{-(ax^2+bx+c)} \overset{x'=x-\varepsilon}{=} \int_0^{+\infty} dx' e^{-(ax'^2+(2a\varepsilon+b)x'+(b\varepsilon+c))},$$

and

$$\int_0^{+\infty} dx e^{-(A_1x^2+A_2x+A_3)} = \frac{1}{2}\sqrt{\frac{\pi}{A_1}}e^{(A_2^2-4A_1A_3)/4A_1}\cdot \mathrm{erfc}(\frac{A_2}{2\sqrt{A_1}}). \tag{3.4.27}$$

By taking $A_1 = a$, $A_2 = 2a\varepsilon + b$ and $A_3 = b\varepsilon + c$ in (3.4.27), we eventually get

$$\int_{\epsilon}^{+\infty} dx e^{-(ax^2+bx+c)} = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/4a}\cdot \mathrm{erfc}(\frac{b}{2\sqrt{a}}+\varepsilon\sqrt{a}). \tag{3.4.28}$$

Now, the first term in the integrand of (3.4.26) can be evaluated as,

$$\int_{\epsilon}^{+\infty} dx e^{-\pi x^2-\pi x(\sqrt{t}+\frac{2y}{\sqrt{t}}+(2m+1)\sqrt{t})} = \frac{1}{2}e^{\pi\frac{((m+1)t+y)^2}{t}}\cdot \mathrm{erfc}\left(\sqrt{\tfrac{\pi}{t}}((m+1)t+y+\sqrt{t}\varepsilon)\right).$$

The second term can be treated similarly and the integration (3.4.26) simplifies to,

$$\phi(it,iy) \overset{t\to\infty}{\asymp} \frac{1}{2}\sum_{m=0}^{\infty} e^{\pi\frac{((m+1)t+y)^2}{t}}\, \mathrm{erfc}\left(\sqrt{\tfrac{\pi}{t}}((m+1)t+y+\sqrt{t}\varepsilon)\right)$$
$$-\frac{1}{2}\sum_{m=0}^{\infty} e^{\pi\frac{(y-mt)^2}{t}}\, \mathrm{erfc}\left(\sqrt{\tfrac{\pi}{t}}(mt-y+\sqrt{t}\varepsilon)\right).$$

We isolate the $m = 0$ term in the first sum (the only term where the argument of erfc is small as $t \to \infty$), rearrange the remaining series, and use the asymptotic expansion,

$$\operatorname{erfc}(z) \overset{z\to\infty}{\asymp} \tfrac{1}{\sqrt{\pi}}\, e^{-z^2} \sum_{n\geq 0} (-1)^n \frac{(2n-1)!!}{2^n\, z^{2n+1}}$$

for large positive z. This gives,

$$\phi(it, iy) \overset{t\to\infty}{\asymp} -\frac{1}{2} e^{\pi\frac{y^2}{t}} \operatorname{erfc}\left(\tfrac{\sqrt{\pi}(\sqrt{t}\varepsilon - y)}{\sqrt{t}}\right)$$
$$+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n-1)!!}{(2\pi)^{n+1}} \left[\frac{e^{-\frac{\pi\varepsilon(2mt+2y+\sqrt{t}\varepsilon)}{\sqrt{t}}}}{(m\sqrt{t} + \frac{y}{\sqrt{t}} + \varepsilon)^{2n+1}} - \frac{e^{-\frac{\pi\varepsilon(2mt-2y+\sqrt{t}\varepsilon)}{\sqrt{t}}}}{(m\sqrt{t} - \frac{y}{\sqrt{t}} + \varepsilon)^{2n+1}} \right]. \quad (3.4.29)$$

We can now set $\varepsilon = 0$, after which the m series can be expressed through the zeta function defined by,

$$\zeta(s, a) = \sum_{m=0}^{\infty} (m + a)^{-s}$$

as,

$$\phi(it, iy) \overset{t\to\infty}{\asymp} -\frac{1}{2}\, e^{\pi\frac{y^2}{t}} \operatorname{erfc}\left(-\sqrt{\tfrac{\pi}{t}}y\right) + \frac{1}{2}\left(\cot\left(\tfrac{\pi y}{t}\right) - \tfrac{t}{\pi y}\right)$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n\,(2n-1)!!}{(2\pi)^{n+1}}\, \frac{\zeta(2n+1, 1+\frac{y}{t}) - \zeta(2n+1, 1-\frac{y}{t})}{t^{n+\frac{1}{2}}}\,.$$

This can be expanded further, using

$$\zeta(n, 1+x) = \sum_{i\geq 0} (-1)^i \binom{n+i-1}{n-1} x^i \zeta(n+i),$$

where $\zeta(s) = \sum_{m\geq 1} m^{-s}$. We thus find,

$$\phi(it, iy) \overset{t\to\infty}{\asymp} -\frac{1}{2}\, e^{\pi\frac{y^2}{t}} \operatorname{erfc}\left(-\sqrt{\tfrac{\pi}{t}}y\right)$$
$$- \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-4)^j \pi^{2j+n+1} B_{2(j+n+1)}}{(j+n+1)\,(2j+1)!\,n!}\, \frac{y^{2j+1}}{t^{\frac{3}{2}+2j+n}}\,, \quad (3.4.30)$$

where the Bernoulli numbers B_{2n} are related to the zeta function values at even positive integers by the following definition,

$$\zeta(2n) = \pi^{2n}\, \frac{2^{n-1}(-1)^{n+1}}{n!\,(2n-1)!!}\, B_{2n}\,.$$

We could also note that the erfc function in (3.4.30) can be decomposed further, with the result [23],

$$\phi(it, iy) \overset{t\to\infty}{\asymp} -\frac{1}{2}\, e^{\pi\frac{y^2}{t}} - \sum_{n=0}^{\infty} \frac{\pi^n B_{2n}}{n!\, t^{\frac{1}{2}+n}}\, y\; {}_1F_1\left(1-n, \tfrac{3}{2}, \tfrac{\pi y^2}{t}\right).$$

We next find the small-t expansion [23]. Writing,

$$\phi(it,iy) =$$
$$= -\frac{1}{2}\int_{-\infty}^{+\infty} dx\, e^{-\pi x^2}\left[\cosh\left(2\pi x\tfrac{y}{\sqrt{t}}\right) + \coth\left(\pi x\sqrt{t}\right)\sinh\left(2\pi x\tfrac{y}{\sqrt{t}}\right)\right], \quad (3.4.31)$$

we readily calculate the cosh integral and expand coth with the result,

$$\phi(it,iy) \overset{t\to 0}{\asymp} -\frac{1}{2}e^{\frac{\pi y^2}{t}} - \sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}\int_0^{\infty} dx\, e^{-\pi x^2}(\pi x\sqrt{t})^{2j-1}\sinh\left(2\pi x\tfrac{y}{\sqrt{t}}\right).$$

This involves the integrals,

$$\int_0^{\infty} dx\, e^{-\pi x^2}x^{-1}\sinh(\beta x) = -\tfrac{i\pi}{2}\,\mathrm{erf}\left(\tfrac{i\beta}{2\sqrt{\pi}}\right),$$
$$\int_0^{\infty} dx\, e^{-\pi x^2}x^{2j-1}\sinh(\beta x) = (-1)^j\frac{i\sqrt{\pi}}{(4\pi)^j}e^{\frac{\beta^2}{4\pi}}H_{2j-1}\left(\tfrac{i\beta}{2\sqrt{\pi}}\right), \quad j\geq 1,$$

where H_m are the Hermite polynomials. They can be written as,

$$H_m(x) = (2x)^m\left(1 - \tbinom{m}{2}\tfrac{1}{2x^2} + 1\cdot 3\tbinom{m}{4}\tfrac{1}{(2x^2)^2} - 1\cdot 3\cdot 5\tbinom{m}{6}\tfrac{1}{(2x^2)^3} + \ldots\right),$$

which gives [23],

$$\phi(it,iy) \overset{t\to 0}{\asymp} -\frac{1}{2}e^{\pi\frac{y^2}{t}} + \tfrac{i}{2\sqrt{t}}\,\mathrm{erf}\left(i\sqrt{\tfrac{\pi}{t}}\,y\right)$$
$$- e^{\pi\frac{y^2}{t}}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{B_{2(j+n+1)}\,4^j\pi^{2j+n+1}}{(j+n+1)\,n!\,(2j+1)!}\,y^{2j+1}\,t^n. \quad (3.4.32)$$

And finally to verify consistency with the S-transform formula (3.4.25) we rewrite it as,

$$\phi\left(\tfrac{i}{t},\tfrac{y}{t}\right) + i\sqrt{t}\,e^{-\pi\frac{y^2}{t}}\,\phi(it,iy) = -\tfrac{i}{2}\sqrt{t} - \frac{1}{2}e^{-\pi\frac{y^2}{t}}.$$

With the above asymptotic expansions (3.4.30) and (3.4.32), we then find,

$$(3.4.32)\Big|_{\substack{t\to\frac{1}{t}\\ y\to -i\frac{y}{t}}} + i\sqrt{t}\,e^{-\pi\frac{y^2}{t}}\cdot(3.4.30) = -\tfrac{i}{2}\sqrt{t} - \frac{1}{2}e^{-\pi\frac{y^2}{t}}.$$

Showing that the asymptotic expansions of integral (3.4.3) obey the S transformation formula (3.4.25).

Chapter 4

Modular transformations of $N = 2$ characters via Appell Functions

4.1 Introduction and basic definitions

Admissible $N = 2$ superconformal characters at central charge $c = 3(1 - \frac{2p}{u})$ with p and u two coprime integers are generically labelled by three discrete integers r, s and θ in the ranges,

$$1 \leq r \leq u-1, \qquad 1-p \leq s \leq p, \qquad \theta \in \mathbb{Z}, \qquad [Ramond] \tag{4.1.1}$$

$$1 \leq r \leq u-1, \qquad 1-p \leq s \leq p, \qquad \theta \in \mathbb{Z}+\frac{1}{2}, \ [Neveu - Schwarz]. \tag{4.1.2}$$

They are functions of two complex variables q and z, $q = e^{2i\pi\tau}$ and $z = e^{2i\pi\nu}$, with $\tau, \nu \in \mathbb{C}$ and $Im(\tau) > 0$. Highest weight characters are usually constructed as series in the variable q, taking into account the existence of null vectors, whose presence in a Verma module indicates that states of zero norm appear and should be removed if one is interested in irreducible representations. The condition $Im(\tau) > 0$ ensures the convergence of such series.

A vast literature is available on $N = 2$ superconformal characters, and we refer to [26] for notations and conventions, as well as for some background on the characters we use here. $N = 2$ characters generically count fermionic as well as bosonic states which might appear in a superconformal field theory with $N = 2$ symmetry, and form the building blocks of torus partition functions of such theories. Unlike bosonic fields, which must be periodic when translated by any of the two periods ω_1, ω_2 of the torus in order to contribute to the construction of a consistent partition function, a fermionic field $\psi(z)$ may pick up a factor of -1. If it does, i.e if for instance,

$$\psi(z+\omega_1) = -\psi(z), \tag{4.1.3}$$

it is said to satisfy Neveu-Schwarz (NS) boundary conditions w.r.t. period ω_1. If it is periodic instead, it is said to satisfy Ramond (R) boundary conditions. Since there are two periods, a fermionic field may be defined according to four types of boundary conditions: (R,R), (R,NS), (NS,R) and (NS,NS). The characters describing states generated by such fermionic fields are respectively called 'super Ramond', 'Ramond', 'super Neveu-Schwarz' and 'Neveu-Schwarz' characters. In the Ramond sector, the character series only have integer powers in the variable q (apart from a possible fractional offset).They are given by,

$$\omega_{(r,s,u,p;\theta)}(q,z) = z^{-\frac{c}{3}\theta} q^{\frac{c}{6}(\theta^2-\theta)} \omega_{(r,s,u,p)}(q, zq^{-\theta}), \tag{4.1.4}$$

where the 'untwisted' (i.e. $\theta = 0$) characters read,

$$\omega_{(r,s,u,p)}(q,z) = z^{s-1-\frac{p}{u}(r-1)} q^{\frac{1}{8}} \frac{\vartheta_{(1,0)}(q,z)}{\tilde{\eta}(q)^3} \varphi_{(r,s,u,p)}(q,z), \tag{4.1.5}$$

with

$$\varphi_{(r,s,u,p)}(q,z) = \sum_{m\in\mathbb{Z}} q^{m^2up-mu(s-1)} \left(\frac{q^{mpr}}{1+z^{-1}q^{mu}} - q^{r(s-1)} \frac{q^{-mpr}}{1+z^{-1}q^{mu-r}} \right). \tag{4.1.6}$$

We therefore write the twisted Ramond non-unitary $N=2$ superconformal characters as,

$$\boxed{\omega_{(r,s,u,p;\theta)}(q,z) = z^{s-1-\frac{p}{u}(r-1-2\theta)} q^{-\frac{p}{u}\theta^2-(s-1-\frac{p}{u}r)\theta} q^{\frac{1}{8}} \frac{\vartheta_{(1,0)}(q,z)}{\tilde{\eta}^3(q)} \varphi_{(r,s,u,p;\theta)}(q,z),} \tag{4.1.7}$$

where we have introduced the following notation,

$$\varphi_{(r,s,u,p;\theta)}(q,z) \equiv \varphi_{(r,s,u,p)}(q, zq^{-\theta}) \tag{4.1.8}$$

and used (2.2.14). To obtain the Neveu-Schwarz characters, one allows the spectral flow, or twist parameter θ to be half-integer. They may be obtained from the

Ramond characters by shifting the angular variable as $z \to z\,q^{\pm\frac{1}{2}}$,

$$\omega_{(r,s,u,p;\theta\mp\frac{1}{2})}(q,z) = z^{\pm\frac{c}{6}}\, q^{\frac{c}{6}[\frac{1}{4}\pm\frac{1}{2}]}\omega_{(r,s,u,p;\theta)}(q,zq^{\pm\frac{1}{2}}). \tag{4.1.9}$$

We choose to define the NS characters as those obtained from thr R characters (4.1.7) by shifting $z \to zq^{-\frac{1}{2}}$, as comparison with the existing literature on *unitary* minimal $N=2$ characters (see [26,32,33]) is straightforward in this case. The supercharacters are obtained by shifting $z \to -z$ in the R and NS characters.

In this chapter, we are particularly interested in the behaviour of $N=2$ characters under the modular transformation $S : \tau \to -\frac{1}{\tau}$ where τ, the torus modulus, is given by the ratio of the two periods, namely $\tau = \frac{\omega_2}{\omega_1}$. Therefore, the S modular transformation essentially interchanges the roles of ω_1 and ω_2 and transforms R characters in super NS characters. We have chosen to study the S transform of the R characters (4.1.7), and we expect to be able to re express the transformed characters as a linear combination of super NS characters of the type

$$\omega_{(r',s',u,p;\theta'-\frac{1}{2})}(q,-z), \tag{4.1.10}$$

possibly up to 'corrective terms'. As we will see, those actually appear whenever the parameter p is not one. The origin of this appearance is rooted in the non-quasiperiodicity of the function $\varphi_{(r,s,u,p;\theta)}(z,q)$ in the spectral flow parameter θ. Indeed using the explicit expression (4.1.6), one can show that,

$$\begin{aligned}\varphi_{(r,s,u,p;un)}(q,z) = {} & z^{-2np}q^{-pn(r-un)+un(s-1)}\,[\,\varphi_{(r,s,u,p)}(q,z)\\ & + \begin{cases} -\displaystyle\sum_{j=0}^{2pn-1}(-1)^j z^{j+1}\Lambda_{(s+j,r+1,u,p)}(q,1)\,], & n\in\mathbb{N}\,,\\ \displaystyle\sum_{j=2pn}^{-1}(-1)^j z^{j+1}\Lambda_{(s+j,r+1,u,p)}(q,1)\,], & n\in-\mathbb{N}\,.\end{cases}\end{aligned} \tag{4.1.11}$$

The derivation of the S modular transformation of series of the φ-type is notoriously difficult in general. A previous attempt at deriving the $N=2$ characters modular transformations in the general case has been made [26]. The method used there takes advantage of the following relation between $\widehat{s\ell}(2)$ admissible characters

$X^{\widehat{s\ell}(2)}_{(r,s,u,p)}(q,z)$ and $N = 2$ superconformal characters (4.1.4) at a common central charge value of $c = \frac{3k}{k+2} = 3(1-\frac{2p}{u})$ [28],

$$X^{\widehat{s\ell}(2)}_{(r,s,u,p)}(q,z)\vartheta_{(1,0)}(q,zy) = \sum_{\theta\in\mathbb{Z}} \omega_{(r,s,u,p;\theta)}(q,y) y^{\frac{2p}{u}(j-\theta)} z^{j-\theta} q^{\frac{p}{u}(j-\theta+\frac{1}{2})^2}.$$

The derivation relies on the knowledge of how $\widehat{s\ell}(2)$ characters transform under the modular group [6]. Our method is more powerful because it works for a whole class of characters whose relation to $\widehat{s\ell}(2)$ characters is not explicitly known. What we exploit is the non-quasiperiodicity nature of the characters. The latter happens to be encoded in the higher-level Appell functions we discussed in the previous chapters. The relation between Appell functions and $N = 2$ admissible characters is through the following formula,

$$\boxed{\begin{aligned}\varphi_{r,s,u,p;\theta}(\tau,\nu) &= \mathcal{K}_{2p}(u\tau, (+\tfrac{r}{2} - \tfrac{(s-1)u}{2p})\tau, \tfrac{1}{2} - \nu - [\tfrac{r}{2} - \tfrac{(s-1)u}{2p} - \theta]\tau) \\ &\quad - e^{2i\pi r(s-1)\tau}\mathcal{K}_{2p}(u\tau, (-\tfrac{r}{2} - \tfrac{(s-1)u}{2p})\tau, \tfrac{1}{2} - \nu - [\tfrac{r}{2} - \tfrac{(s-1)u}{2p} - \theta]\tau)\,. \quad (4.1.12)\end{aligned}}$$

The rest of this chapter consists in using the S modular properties of Appell functions derived in Chapter 3 in order to provide the S transform of non-unitary $N = 2$ superconformal characters.

4.2 S transform of $N = 2$ non-unitary admissible characters

4.2.1 Modular transformation of $\varphi_{r,s,u,p;\theta}(\tau,\nu)$

In view of (2.2.19) and (2.2.24), as well as of the definition of characters (4.1.7), it is clear that the non-trivial derivation is the S transform of (4.1.12), which we write

$$\begin{aligned}\varphi_{r,s,u,p;\theta}(\tfrac{-1}{\tau},\tfrac{\nu}{\tau}) &= \mathcal{K}_{2p}(\tfrac{-u}{\tau}, (\tfrac{-r}{2} + \tfrac{(s-1)u}{2p})/\tau, [\tfrac{\tau}{2} - \nu + \tfrac{r}{2} - \tfrac{(s-1)u}{2p} - \theta]/\tau) \\ &\quad - e^{-2i\pi\frac{r(s-1)}{\tau}}\mathcal{K}_{2p}(\tfrac{-u}{\tau}, (\tfrac{+r}{2} + \tfrac{(s-1)u}{2p})/\tau, [\tfrac{\tau}{2} - \nu + \tfrac{r}{2} - \tfrac{(s-1)u}{2p} - \theta]/\tau)\,. \quad (4.2.1)\end{aligned}$$

In physical applications, and in particular when one is concerned with the construction of modular invariant partition functions, it is extremely valuable to re express (4.2.1) in terms of $\varphi_{r,s,u,p;\theta}(\tau,\nu)$ or more exactly in terms of a linear combination of functions $\varphi_{r',s',u,p;\theta'}(\tau,\nu)$. In general, such a combination appears as a 'leading term' in (4.2.1), but the non-quasiperiodicity generates corrective terms whose structure is worth studying too.

Step 1: Identification of the 'leading term'

- Using the essential formula (3.4.21) giving the S transform of higher-level Appell functions, the two level $2p$ Appell functions entering in the expression (4.2.1) become,

$$\mathcal{K}_{2p}\Big(\tfrac{-1}{\tau/u},\frac{\frac{\pm r}{2u}+\frac{s-1}{2p}}{\tau/u},\frac{\tilde\mu-\frac{s-1}{2p}}{\tau/u}\Big)=\frac{\tau}{u}\,\mathcal{C}_{\pm r,s,\theta}(\tau,\nu)\Big[\mathcal{K}_{2p}\big(\tfrac{\tau}{u},\tfrac{\pm r}{2u}+\tfrac{s-1}{2p},\tilde\mu-\tfrac{s-1}{2p}\big)$$
$$+\frac{1}{2p}\sum_{a=0}^{2p-1}e^{\frac{2i\pi}{\tau}up(\frac{a}{2p}+\tilde\mu-\frac{s-1}{2p})^2}\Phi\big(\tfrac{\tau}{2pu},\tilde\mu-\tfrac{s-1}{2p}+\tfrac{a}{2p}\big)\vartheta\big(\tfrac{\tau}{2pu},\tfrac{\pm r}{2u}+\tfrac{s-1}{2p}-\tfrac{a}{2p}\big)\Big],\quad (4.2.2)$$

where

$$\tilde\mu = \tfrac{1}{u}\big(\tfrac{\tau}{2}-\nu+\tfrac{r}{2}-\theta\big),\quad (4.2.3)$$
$$\mathcal{C}_{-r,s,\theta}(\tau,\nu) = e^{\frac{2i\pi}{\tau}up[(\frac{-r}{2u}+\frac{s-1}{2p})^2-(\tilde\mu-\frac{s-1}{2p})^2]},\quad (4.2.4)$$
$$\mathcal{C}_{r,s,\theta}(\tau,\nu) = e^{2i\pi\frac{r(s-1)}{\tau}}\mathcal{C}_{-r,s,\theta}(\tau,\nu).\quad (4.2.5)$$

- Now using the identity (2.4.4) to cancel the terms $\frac{s-1}{2p}$ appearing in both $\mathcal{K}_{2p}$'s in (4.2.2), one arrives at the following expression for φ:

$$\boxed{\varphi_{r,s,u,p;\theta}\big(\tfrac{-1}{\tau},\tfrac{\nu}{\tau}\big)=\mathcal{G}+\mathcal{H}\ ,\qquad (4.2.6)}$$

where,

$$\mathcal{G}=\frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)\big[\mathcal{K}_{2p}\big(\tfrac{\tau}{u},\tfrac{-r}{2u},\tilde\mu\big)-\mathcal{K}_{2p}\big(\tfrac{\tau}{u},\tfrac{\pm r}{2u},\tilde\mu\big)\big],\quad (4.2.7)$$

and,

$$\mathcal{H} = \frac{\tau}{2pu} \mathcal{C}_{-r,s,\theta}(\tau,\nu) \sum_{a=0}^{2p-1} e^{\frac{2i\pi}{\tau} up(\frac{a}{2p}+\tilde{\mu}-\frac{s-1}{2p})^2} \Phi(\tfrac{\tau}{2pu}, \tilde{\mu} - \tfrac{s-1}{2p} + \tfrac{a}{2p})$$
$$\times \left[\vartheta(\tfrac{\tau}{2pu}, \tfrac{-r}{2u} + \tfrac{s-1}{2p} - \tfrac{a}{2p}) - \vartheta(\tfrac{\tau}{2pu}, \tfrac{+r}{2u} + \tfrac{s-1}{2p} - \tfrac{a}{2p})\right]. \quad (4.2.8)$$

The $\mathcal{H}$ term is one of the corrective terms mentioned above and will be improved in Step 2 below. We continue with the identification of the 'leading term' emerging from the term $\mathcal{G}$.

• In order for $\mathcal{G}$ to contain a leading term, i.e. to contain functions $\varphi_{r',s',u,p;\theta'}(\tau,\nu)$, the Appell functions in (4.2.7) should have $u\tau$ as first argument instead of $\frac{\tau}{u}$. We therefore use the crucial period-increasing formula (2.4.21) derived in Chapter 2, in which $q \to q^{\frac{1}{u}}$. We also insert

$$x = e^{-2i\pi\frac{\tau}{2u}}, \quad (4.2.9)$$

and

$$y = e^{2i\pi\tilde{\mu}} = e^{\frac{2i\pi}{u}(\frac{\tau}{2}-\nu+\frac{r}{2}-\theta)} = x^{-1} z^{-\frac{1}{u}} q^{\frac{1}{2u}} e^{-\frac{2i\pi\theta}{u}}. \quad (4.2.10)$$

Note that the variables x and y are not independent in the case of $N = 2$, as expected since the Kac-Moody subalgebra is $U(1)$. The $\mathcal{G}$ term becomes,

$$\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2, \quad (4.2.11)$$

where,

$$\mathcal{G}_1 = \frac{\tau}{u} \mathcal{C}_{-r,s,\theta}(\tau,\nu) \sum_{r'=0}^{u-1}\sum_{b=1}^{u} q^{\frac{p}{2u}(r'+2b)-\frac{pb}{u}(r'+b)} z^{-\frac{p}{u}(r'+2b)} e^{2i\pi\frac{p}{u}(br-\theta(r'+2b))}$$
$$\times \left[\mathcal{K}_{(2p)}(q^u, e^{-i\pi r} q^{\frac{r'}{2}}, e^{i\pi r} e^{2i\pi(\frac{\tau}{2}-\nu-\theta)} q^{-\frac{r'}{2}-b})\right.$$
$$\left. - \mathcal{K}_{(2p)}(q^u, e^{+i\pi r} q^{-\frac{r'}{2}}, e^{i\pi r} e^{2i\pi(\frac{\tau}{2}-\nu-\theta)} q^{-\frac{r'}{2}-b})\right],$$
$$= \frac{\tau}{u} \mathcal{C}_{-r,s,\theta}(\tau,\nu) \sum_{r'=0}^{u-1}\sum_{b=1}^{u} q^{\frac{p}{2u}(r'+2b)-\frac{pb}{u}(r'+b)} z^{-\frac{p}{u}(r'+2b)} e^{2i\pi\frac{p}{u}(br-\theta(r'+2b))}$$
$$\times \left[\mathcal{K}_{(2p)}(q^u, q^{\frac{r'}{2}}, q^{\frac{1}{2}} z^{-1} q^{-\frac{r'}{2}-b})\right.$$
$$\left. - \mathcal{K}_{(2p)}(q^u, q^{-\frac{r'}{2}}, q^{\frac{1}{2}} z^{-1} q^{-\frac{r'}{2}-b})\right], \text{ using}(2.4.6) \quad (4.2.12)$$

and,

$$\mathcal{G}_2 = \frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)\sum_{b=1-u}^{u}\sum_{\substack{s'=1 \\ s'u-r'p<2pb}}^{2p-1}\sum_{r'=0}^{u-1} q^{\frac{p}{2u}(r'+2b)-\frac{pb}{u}(r'+b)+bs'-\frac{s'}{2}} z^{-\frac{p}{u}(r'+2b)+s'} e^{2i\pi\frac{p}{u}(br-\theta(r'+2b))}\times$$

$$\times\, \Lambda_{s',r'+1,u,p}(\tau,0). \quad (4.2.13)$$

The $\mathcal{G}_2$ term is another corrective term and we delay its treatment until Step 2. We continue the discussion of $\mathcal{G}_1$.

- A quick look at (4.1.12) shows that $\mathcal{G}_1$ can be rewritten

$$\mathcal{G}_1 = \frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)\sum_{r'=1}^{u-1}\sum_{b=1}^{u} q^{\frac{p}{2u}(r'+2b)-\frac{pb}{u}(r'+b)} z^{-\frac{p}{u}(r'+2b)} e^{2i\pi\frac{p}{u}(br-\theta(r'+2b))}$$

$$\times\, \varphi_{r',1,u,p;-b+\frac{1}{2}}(\tau,\nu+\tfrac{1}{2}), \quad (4.2.14)$$

where we have taken into account that the term $r'=0$ in (4.2.12) does not contribute. This is a perfectly reasonable 'leading term' as it is a linear combination of φ functions of the type (4.1.12). Note however that this linear combination is not 'democratic', i.e. although it sweeps all values of r' in the range $1\le r'\le u-1$, it singles out the value $s'=1$ when a priori that parameter takes values in the range $1-p\le s'\le p$ (recall (4.1.1)) Furthermore, we argued in the previous section that the transformed Ramond characters should be re expressed in terms of super Neveu-Schwarz characters (4.1.10), where θ' is meant to run over the integers $1,..,u$. This would not be the case if we kept the twist in (4.2.14) to be $-b+\frac{1}{2}$ where $b=1,...,u$. **In the rest of this thesis, we choose to express** (4.2.14) **in terms of** $\varphi_{r',s,u,p;b-\frac{1}{2}}(\tau,\nu+\frac{1}{2})$ **instead of** $\varphi_{r',1,u,p;-b+\frac{1}{2}}(\tau,\nu+\frac{1}{2})$ **when studying the modular transformations of a given** $N=2$ **character** $\omega_{r,s,u,p;\theta}(\tau,\nu)$**. We also assume, without loss of generality, that** $1\le s\le p$. In order to single out $s'=s$, we use the Appell functions periodicity property (2.4.11), with $p\to 2p$, $q\to q^u$, $n=s-1$, $y=q^{\frac{1}{2}}z^{-1}q^{-\frac{r'}{2}-b}$ and $x\to q^{\frac{\pm r'}{2}}$ and write,

$$\mathcal{K}_{(2p)}(q^u, q^{\frac{\pm r'}{2}}, q^{\frac{1}{2}}z^{-1}q^{-\frac{r'}{2}-b}) =$$
$$(z^{-1}q^{\frac{1}{2}(1\pm r'-r'-2b)})^{1-s}\,\mathcal{K}_{(2p)}(q^u, q^{\frac{\pm r'}{2}-\frac{(s-1)u}{2p}}, q^{\frac{1}{2}}z^{-1}q^{-\frac{r'}{2}+\frac{(s-1)u}{2p}-b})$$
$$-\sum_{j=1}^{s-1}(z^{-1}q^{\frac{1}{2}(1\pm r'-r'-2b)})^{-j}\theta(q^{2up}, q^{\pm r'p-ju}). \qquad (4.2.15)$$

We then insert (4.2.15) in (4.2.12) (remembering that the term $r'=0$ does not contribute) and eventually obtain,

$$\mathcal{G}_1 = \mathcal{L} + \mathcal{L}_s\ , \qquad (4.2.16)$$

where

$$\mathcal{L} = \frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)\sum_{r'=1}^{u-1}\sum_{b=1}^{u} q^{-\frac{p}{u}(b^2+br'-b-\frac{r'}{2})+(\frac{1}{2}-b)(1-s)}z^{-\frac{p}{u}(r'+2b)+s-1}e^{-2i\pi\frac{p}{u}(r'+2b)\theta+2i\pi\frac{pbr}{u}}$$
$$\times\,\varphi_{r',s,u,p;-b+\frac{1}{2}}(\tau,\nu+\tfrac{1}{2}), \qquad (4.2.17)$$

and

$$\mathcal{L}_s = -\frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)\sum_{r'=1}^{u-1}\sum_{b=1}^{u}\sum_{j=1}^{s-1} q^{-\frac{p}{u}(b^2+br'-b-\frac{r'}{2})-(\frac{1}{2}-b)j}z^{-\frac{p}{u}(r'+2b)+j}e^{-2i\pi\frac{p}{u}(r'+2b)\theta+2i\pi\frac{pbr}{u}}$$
$$\times\,\Lambda_{j,r'+1,u,p}(\tau,0). \qquad (4.2.18)$$

Finally we relabel $b\to -b+u+1$ in (4.2.17) and write,

$$\mathcal{L} = \frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)$$
$$\times\sum_{r'=1}^{u-1}\sum_{b=1}^{u} q^{\frac{p}{2u}(r'-2b+2(u+1))-\frac{p(-b+u+1)}{u}(r'-b+u+1)+(b-u-\frac{1}{2})(1-s)}z^{-\frac{p}{u}(r'-2b+2(u+1))+s-1}$$
$$\times\, e^{2i\pi\frac{p}{u}(r-br-\theta(r'-2b+2))}\varphi_{r',s,u,p;b-\frac{1}{2}-u}(\tau,\nu+\tfrac{1}{2}). \qquad (4.2.19)$$

Using the periodicity property (2.4.9), we have

$$\varphi_{r',s,u,p;b-\frac{1}{2}-u}(\tau,\nu+\tfrac{1}{2}) = z^{2p}q^{p(r'-2b+u+1)-u(s-1)}\,[\,\varphi_{r',s,u,p;b-\frac{1}{2}}(\tau,\nu+\tfrac{1}{2})$$
$$-\sum_{j=0}^{2p-1} z^{-j}q^{(b-\frac{1}{2})j}\Lambda_{(j-s+1,s,u,p)}(q,q^{r'+s-1})\,],$$
$$= z^{2p}q^{p(r'-2b+u+1)-u(s-1)}\,[\,\varphi_{r',s,u,p;b-\frac{1}{2}}(\tau,\nu+\tfrac{1}{2})$$
$$-\sum_{j=0}^{2p-1} z^{-j}q^{(b-\frac{1}{2})j}\Lambda_{s-j-1,r'+1,u,p}(\tau,0)\,], \qquad (4.2.20)$$

so that, after using (4.2.4), we arrive at

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_c \ , \tag{4.2.21}$$

where

$$\begin{aligned}\mathcal{L}_1 = \frac{\tau}{u}\, & e^{-2i\pi\frac{\nu}{\tau}[s-1-\frac{p}{u}(r-2\theta)]} e^{-2i\pi\frac{\theta}{\tau}[s-1-\frac{p}{u}r]}\, e^{-2i\pi\frac{p}{u}\tau[\nu^2-\theta^2]} \times \\ & \times \sum_{r'=1}^{u-1}\sum_{b=1}^{u} q^{-\frac{p}{2u}(r'-2b+2b^2-2br'+\frac{1}{2})-(b-\frac{1}{2})(s-1)} z^{-\frac{p}{u}(r'-2b+1)+s-1} \\ & \times e^{i\pi\frac{p}{u}[(s-1)+(r-2br-2\theta(r'-2b+1))}\, \varphi_{r',s,u,p;b-\frac{1}{2}}(\tau,\nu+\tfrac{1}{2})\end{aligned} \tag{4.2.22}$$

and we have generated yet another corrective term,

$$\begin{aligned}\mathcal{L}_c = -\frac{\tau}{u}\, & e^{-2i\pi\frac{\nu}{\tau}[s-1-\frac{p}{u}(r-2\theta)]} e^{-2i\pi\frac{\theta}{\tau}[s-1-\frac{p}{u}r]}\, e^{-2i\pi\frac{p}{u}\tau[\nu^2-\theta^2]} \times \\ & \times \sum_{r'=1}^{u-1}\sum_{b=1}^{u} q^{-\frac{p}{2u}(r'-2b+2b^2-2br'+\frac{1}{2})-(b-\frac{1}{2})(s-1)} z^{-\frac{p}{u}(r'-2b+1)+s-1} \\ & \times e^{i\pi\frac{p}{u}[(s-1)+(r-2br-2\theta(r'-2b+1))} \sum_{j=0}^{2p-1} z^{-j} q^{(b-\frac{1}{2})j} \Lambda_{s-j-1,r'+1,u,p}(\tau,0)\,].\end{aligned} \tag{4.2.23}$$

Therefore, at this point, the S transform of $\varphi_{r,s,u,p;\theta}(\tau,\nu)$ reads,

$$\boxed{\varphi_{r,s,u,p;\theta}(\tfrac{-1}{\tau},\tfrac{\nu}{\tau}) = \mathcal{L}_1 + \mathcal{L}_c + \mathcal{L}_s + \mathcal{G}_2 + \mathcal{H},} \tag{4.2.24}$$

where the five terms are respectively given by (4.2.22), (4.2.23), (4.2.18), (4.2.13) and (4.2.8).

This completes Step 1. We now proceed to rearrange all the corrective terms in the most compact possible way.

Step 2: the corrective terms $\mathcal{L}_c + \mathcal{L}_s + \mathcal{G}_2 + \mathcal{H}$

- We start with the term $\mathcal{H}$ containing the complicated Φ functions. We define some new variables to make the calculations as concise as they can be, namely we set

$$\frac{\gamma}{u} = \tilde{\mu} - \frac{s-1}{2p} \tag{4.2.25}$$

$$\frac{\eta^-}{u} = -\frac{r}{2u} + \frac{s-1}{2p} \tag{4.2.26}$$

$$\frac{\eta^+}{u} = +\frac{r}{2u} + \frac{s-1}{2p} \,, \tag{4.2.27}$$

with $\tilde{\mu}$ given in (4.2.3). Using these new variables, $\mathcal{H}$ simplifies to,

$$\mathcal{H} = \mathcal{H}^- - \mathcal{H}^+ \,, \tag{4.2.28}$$

where,

$$\mathcal{H}^\pm = \frac{\tau}{2pu} \mathcal{C}_{-r,s,\theta}(\tau,\nu) e^{2i\pi \frac{p}{u\tau}\gamma^2} \sum_{a=0}^{2p-1} e^{i\pi \frac{u}{2p\tau}a^2 + 2i\pi\frac{\gamma}{\tau}a} \Phi(\tfrac{\tau}{2pu}, \tfrac{\gamma}{u} + \tfrac{a}{2p}) \vartheta(\tfrac{\tau}{2pu}, \tfrac{\eta^\pm}{u} - \tfrac{a}{2p}). \tag{4.2.29}$$

We now perform the calculations for $\mathcal{H}^-$ and then use the final result to express $\mathcal{H}^+$. We split the term $\mathcal{H}^-$ (and then $\mathcal{H}^+$) into two simpler parts, with the hope these will combine with some of the other corrective terms emerging from $\mathcal{G}$. To our knowledge the only way to do that is to use period-increasing formulas for the Φ and the ϑ functions appearing in $\mathcal{H}^-$.

- We rescale Φ by changing $\frac{\mu}{2pu} \to \frac{\gamma}{u} + \frac{a}{2p}$ in (3.4.13) and write,

$$\begin{aligned} \Phi(\tfrac{\tau}{2pu}, \tfrac{\gamma}{u} + \tfrac{a}{2p}) &= \sum_{s'=1}^{2p}\sum_{r'=1}^{u} \Phi(2pu\tau, 2p\gamma + au - u(s'-1)\tau - p(r'-1)\tau) + \mathcal{E}_1 \,, \\ \mathcal{E}_1 &= -\sum_{\substack{s'=1 \\ s'u-r'p>0}}^{p-1}\sum_{r'=1}^{u-1} e^{-i\pi\frac{\tau}{2pu}(\frac{2p\gamma+au}{\tau} - s'u + r'p)^2}. \end{aligned} \tag{4.2.30}$$

The next step is to take the term "au" out of Φ. For that we need the first identity mentioned in (3.4.7). In fact, taking $m = au$, and changing $\mu \to 2p\gamma - u(s'-1)\tau - p(r'-1)\tau$ in this identity gives,

$$\begin{aligned} \Phi(\tfrac{\tau}{2pu}, \tfrac{\gamma}{u} + \tfrac{a}{2p}) = \sum_{s'=1}^{2p}\sum_{r'=1}^{u} & e^{-i\pi\frac{ua^2}{2p\tau} - 2i\pi a \frac{2p\gamma - u(s'-1)\tau - p(r'-1)\tau}{2p\tau}} \\ & \times \Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau) + \mathcal{E}_1 + \mathcal{E}_2 \,, \end{aligned} \tag{4.2.31}$$

where $\mathcal{E}_2$ denotes the exponential terms following from (3.4.7). This term vanishes. Indeed, write

$$\mathcal{E}_2 = \frac{i}{\sqrt{-2pui\tau}} \sum_{s'=1}^{2p}\sum_{r'=1}^{u}\sum_{j=1}^{au} e^{i\pi\frac{j(j-2ua)}{2pu\tau} - 2i\pi j\frac{2p\gamma - u(s'-1)\tau - p(r'-1)\tau}{2pu\tau}} =$$

$$= \frac{i}{\sqrt{-2pui\tau}} \sum_{j=1}^{au} e^{i\pi\frac{j(j-2ua)}{2pu\tau} - 2i\pi j\frac{\gamma}{u\tau}} \sum_{s'=0}^{2p-1}\sum_{r'=0}^{u-1} e^{2i\pi j\frac{s'u+r'p}{2pu}}. \quad (4.2.32)$$

The s' summation gives zero unless $j = 2pn$ for $n \in \mathbb{Z}$. Restricting the sum over j to these values, we then see that the r' sum vanishes unless $n = ku$, for $k \in \mathbb{Z}$, but this condition is incompatible with the ranges $1 \leq j \leq au$ and $0 \leq a \leq 2p - 1$ and therefore the sum above is zero.

• Next using (2.4.13), we represent the theta function in the right hand side of $\mathcal{H}^-$ as,

$$\vartheta(\tfrac{\tau}{2pu}, \tfrac{\eta^-}{u} - \tfrac{a}{2p}) = \sum_{s''=1}^{2p}\sum_{r''=1}^{u} e^{2i\pi(\frac{\eta^-}{u} - \frac{a}{2p})[u(s''-1)+p(r''-1)] + i\pi\frac{u\tau}{2p}[s''-1+\frac{p}{u}(r''-1)]^2}$$
$$\times\, \vartheta(2pu\tau, 2p\eta^- + u(s''-1)\tau + p(r''-1)\tau). \quad (4.2.33)$$

Although the above formula does not seem a familiar kind of period increasing statement for theta functions, it is actually of the proper format to suit our purpose. Appendix D.1 shows how this identity is proven and how we can change the signs of labels inside it. Inserting the rescaled Φ mentioned in (4.2.31) and the period increased ϑ in (4.2.33) into (4.2.29) for $\mathcal{H}^-$, we obtain,

$$\mathcal{H}^- = \mathcal{H}_1^- + \mathcal{H}_2^- \,, \quad (4.2.34)$$

where,

$$\mathcal{H}_1^- = \frac{\tau}{2pu} \mathcal{C}_{-r,s,\theta}(\tau,\nu) e^{2i\pi\frac{p}{u\tau}\gamma^2} \sum_{s'=1}^{2p}\sum_{r'=1}^{u}\sum_{s''=1}^{2p}\sum_{r''=1}^{u} e^{2i\pi(s''-1+\frac{p}{u}(r''-1))\eta^- + i\pi\frac{u\tau}{2p}(s''-1+\frac{p}{u}(r''-1))^2}$$
$$\times\, \Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau)$$
$$\times\, \vartheta(2pu\tau, 2p\eta^- + u(s''-1)\tau + p(r''-1)\tau)$$
$$\times \sum_{a=0}^{2p-1} e^{2i\pi\frac{a}{2p}[u(s'-s'')+p(r'-r'')]}, \quad (4.2.35)$$

and,

$$\mathcal{H}_2^- = \frac{\tau}{2pu}\,\mathcal{C}_{-r,s,\theta}(\tau,\nu)e^{2i\pi\frac{p}{u\tau}\gamma^2}\times$$
$$\times\sum_{a=0}^{2p-1}\sum_{s''=1}^{2p}\sum_{r''=1}^{u} e^{i\pi\frac{ua^2}{2p\tau}+2i\pi a\frac{\gamma}{\tau}+2i\pi(\frac{\eta^-}{u}-\frac{a}{2p})[u(s''-1)+p(r''-1)]+i\pi\frac{u\tau}{2p}[s''-1+\frac{p}{u}(r''-1)]^2}\times$$
$$\times\,\mathcal{E}_1\,\vartheta(2pu\tau, 2p\eta^- + u(s''-1)\tau + p(r''-1)\tau). \tag{4.2.36}$$

• We first concentrate on (4.2.35) and see how it can be simplified. First in (4.2.35) the sum over "a" vanishes unless,

$$\frac{u(s'-s'')+p(r'-r'')}{2p} = n, \qquad n\in\mathbb{Z}\,. \tag{4.2.37}$$

Now based on appendix D.1, if we change

$$\sum_{s''=1}^{2p}\sum_{r''=1}^{u}\to\sum_{s''=1}^{p}\sum_{r''=1}^{2u} \tag{4.2.38}$$

and,

$$\sum_{s'=1}^{2p}\sum_{r'=1}^{u}\to\sum_{s'=1}^{p}\sum_{r'=1}^{2u} \tag{4.2.39}$$

and implementing $(u,p)=1$ one obtains,

$$s'-s''=0 \;\to\; s'=s''\,, \tag{4.2.40}$$

$$r'-r''=2n \;\to\; r''=r'-2n\,, \tag{4.2.41}$$

by which we easily end up with,

$$\sum_{a=0}^{2p-1} e^{2i\pi\frac{a}{2p}[u(s'-s'')+p(r'-r'')]} = 2p\,. \tag{4.2.42}$$

Therefore imposing the conditions (4.2.40) and (4.2.41) in (4.2.35), reduces the sum over "a" to an overall single factor $2p$ and gets rid of the summation on r'' and s''. But now the question is: how many integers "n" satisfy the constraint (4.2.41) ? In order to answer the question, recall that, once we use (4.2.38), the parameter r'' is in the range $1\le r''\le 2u$, and therefore using (4.2.41) we get the constraint of "n" as,

$$1\le r'-2n\le 2u\,. \tag{4.2.43}$$

Hence by bringing the upper limits of r' and s' back again (opposite to (4.2.39)), changing $n \to -n$ and putting everything together we arrive at,

$$\mathcal{H}_1^- = \frac{\tau}{u}\,\mathcal{C}_{-r,s,\theta}(\tau,\nu)e^{2i\pi\frac{p}{u\tau}\gamma^2}$$
$$\times \sum_{s'=1}^{2p}\sum_{r'=1}^{u}\sum_{\substack{n\in\mathbb{Z}\\ 1\le r'+2n\le 2u}} e^{2i\pi\eta^-((s'-1)+\frac{p}{u}(r'+2n-1))+i\pi\frac{\tau}{2pu}(u(s'-1)+p(r'+2n-1))^2}$$
$$\times\, \Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau)$$
$$\times\, \vartheta(2pu\tau, 2p\eta^- + u(s'-1)\tau + p(r'+2n-1)\tau). \tag{4.2.44}$$

• We need now to investigate the term $\mathcal{H}_2^-$. To do that, we first insert the term $\mathcal{E}_1$ appearing in (4.2.30) in (4.2.36). After some simplifications we obtain,

$$\mathcal{H}_2^- = -\frac{\tau}{2pu}\,\mathcal{C}_{-r,s,\theta}(\tau,\nu)\sum_{\substack{s'=1\\ }}^{p-1}\sum_{\substack{r'=1\\ s'u-r'p>0}}^{u-1}\sum_{s''=1}^{2p}\sum_{r'=1}^{u} e^{2i\pi\frac{\gamma}{u}(s'u-r'p)-i\pi\frac{\tau}{2up}(s'u-r'p)^2}$$
$$\times\, e^{2i\pi\eta^-(s''-1+\frac{p}{u}(r''-1))+i\pi\frac{u\tau}{2p}[s''-1+\frac{p}{u}(r''-1)]^2}$$
$$\times\, \vartheta(2pu\tau, 2p\eta^- + u(s''-1)\tau + p(r''-1)\tau)$$
$$\times \sum_{a=0}^{2p-1} e^{2i\pi\frac{a}{2p}[u(s'-s''+1)-p(r'+r''-1)]}\,. \tag{4.2.45}$$

Here also, we do same as (4.2.38) to swap the upper limits of s'' and r'' summations. Now, the sum over "a" vanishes unless,

$$\frac{u(s'-s''+1)-p(r'+r''-1)}{2p} = n, \qquad n\in\mathbb{Z}\,. \tag{4.2.46}$$

Given the new ranges of s'' and r'', and the fact $(u,p)=1$, we have a non-zero sum over "a" when,

$$s'-s''+1=0 \;\to\; s''=s'+1\,, \tag{4.2.47}$$
$$r'+r''-1=-2n \;\to\; r''=-r'-2n+1\,, \tag{4.2.48}$$

and the sum over "a" again yields a factor of $2p$. Since $1\le r''\le 2u$, the integer "n" is constrained to be in the interval,

$$1\le -r'-2n+1\le 2u \;\to\; 1-2u\le r'+2n\le 0\,. \tag{4.2.49}$$

We finally arrive at,

$$\mathcal{H}_2^- = -\frac{\tau}{u} \mathcal{C}_{-r,s,\theta}(\tau,\nu) \sum_{s'=1}^{p-1} \sum_{r'=1}^{u-1} \sum_{\substack{n\in\mathbb{Z} \\ s'u-r'p>0 \\ 1-2u\leq r'+2n\leq 0}} e^{2i\pi\eta^-(s'-\frac{p}{u}(r'+2n))+2i\pi\gamma(s'-\frac{p}{u}r')}$$

$$\times\, e^{-2i\pi n\tau(s'-\frac{p}{u}(r'+n))}\vartheta(2pu\tau, 2p\eta^- + s'u\tau - p(r'+2n)\tau). \quad (4.2.50)$$

So far, we have

$$\boxed{\mathcal{H} = (\mathcal{H}_1^- - \mathcal{H}_1^+) + (\mathcal{H}_2^- - \mathcal{H}_2^+) \equiv \Delta\mathcal{H}_1 + \Delta\mathcal{H}_2 \qquad (4.2.51)}$$

where $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$ are respectively given by (4.2.44) and (4.2.50), and their '+' counterparts are obtained by replacing η^- with η^+. In order to make the corrective terms as compact as possible, we always try to express them in terms of Λ functions (C.2.1).

• We start with $\Delta\mathcal{H}_1 = \mathcal{H}_1^- - \mathcal{H}_1^+$. Our first step is to **isolate** the Φ function in $\mathcal{H}_1^\pm$ in order to produce Λ terms. We use the identity (D.2.8) to deal with the double sum $\sum_{r'=1}^{u}\sum_{n\in\mathbb{Z}}$ in the above expression. What plays the role of $\Psi(r')$ is $\Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau)$ here. We therefore write $\mathcal{H}_1^-$ as,

$$\mathcal{H}_1^- = \frac{\tau}{u} \mathcal{C}_{-r,s,\theta}(\tau,\nu) e^{2i\pi\frac{p}{u\tau}\gamma^2} \times$$

$$\times \sum_{s'=1}^{2p} \sum_{r'=1}^{u} \Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau) \quad (4.2.52)$$

$$\times \sum_{b=1}^{u} e^{2i\pi\eta^-((s'-1)+\frac{p}{u}(2b-[r']_2-1))+i\pi\frac{\tau}{2pu}(u(s'-1)+p(2b-[r']_2-1))^2}$$

$$\times\, \vartheta(2pu\tau, 2p\eta^- + u(s'-1)\tau + p(2b - [r']_2 - 1)\tau)\ ,$$

As we mentioned before, the term $\mathcal{H}^+$ can be obtained by replacing η^- with η^+ in $\mathcal{H}_1^-$ as it is given in (4.2.52). So doing that for equality above, and a change of

$b \to u - b + 1$ simultaneously gives,

$$\begin{aligned}\mathcal{H}_1^+ &= \frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)e^{2i\pi\frac{p}{u\tau}\gamma^2}\times \\ &\times\sum_{s'=1}^{2p}\sum_{r'=1}^{u}\Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau) \\ &\times\sum_{b=1}^{u} e^{2i\pi\eta^+((s'-1)+\frac{p}{u}(2u-2b-[r']_2+1))+i\pi\frac{\tau}{2pu}(u(s'-1)+p(2u-2b-[r']_2+1))^2} \\ &\times\vartheta(2pu\tau, 2p\eta^+ + u(s'-1)\tau + p(2u-2b-[r']_2+1)\tau)\ .\end{aligned} \tag{4.2.53}$$

We now use (2.2.7) to eliminate the term $2pu\tau$ in the second argument of the theta function, Then in the resulting formula in accordance to (D.2.10) and (D.2.11) we may choose to flip the sign of $[r']_2$ in the resulting formula. Thus (4.2.53) is eventually rewritten as,

$$\begin{aligned}\mathcal{H}_1^+ = \frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)e^{2i\pi\frac{p}{u\tau}\gamma^2} &\sum_{s'=1}^{2p}\sum_{r'=1}^{u}\Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau) \\ &\times\sum_{b=1}^{u} e^{2i\pi\eta^+((s'-1)-\frac{p}{u}(2b-[r']_2-1))+i\pi\frac{\tau}{2pu}(u(s'-1)-p(2b-[r']_2-1))^2} \\ &\times\vartheta(2pu\tau, 2p\eta^+ + u(s'-1)\tau - p(2b-[r']_2-1)\tau)\ ,\end{aligned} \tag{4.2.54}$$

At this point we wish to make three remarks. Firstly the Φ functions appearing in $\mathcal{H}_1^-$ and $\mathcal{H}_1^+$ are identical and we can factorise it in the difference $\mathcal{H}_1^- - \mathcal{H}_1^+$. Secondly we have succeeded in changing the sign of all occurrences of $(2b - [r']_2 - 1)$ in the formula above so that, when $\mathcal{H}_1^+$ is subtracted from $\mathcal{H}_1^-$, Λ function is produced. Thirdly we also note that the crucial change of sign for $(2b - [r']_2 - 1)$ could be also obtained by using the flexibility of change of signs available when dealing with theta functions, see appendix D.1. Now we call the amounts of η^+ and η^- from (4.2.26) and (4.2.27) and subtract (4.2.54) from (4.2.52). We write and simplify as,

$$
\begin{aligned}
&\Delta\mathcal{H}_1 = \mathcal{H}_1^- - \mathcal{H}_1^+ = \\
&\frac{\tau}{u}\,\mathcal{C}_{-r,s,\theta}(\tau,\nu)e^{2i\pi\frac{p}{u\tau}\gamma^2}\sum_{s'=1}^{2p}\sum_{r'=1}^{u}\sum_{b=1}^{u}\Phi(2pu\tau, 2p\gamma-u(s'-1)\tau-p(r'-1)\tau) \\
&\qquad\qquad \times e^{i\pi(r+\frac{u}{p}(s-1))((s'-1)-\frac{p}{u}(2b-[r']_2-1))+i\pi\frac{\tau}{2pu}(u(s'-1)-p(2b-[r']_2-1))^2} \\
&\qquad\qquad \times \Lambda_{s'-1,2b-[r']_2,u,p}(\tau,0)\ .
\end{aligned}
\tag{4.2.55}
$$

• We now wish to eliminate any numerical phase appearing in the second argument of Φ. Such phase originates in the γ term (4.2.25), and the Φ function is given by,

$$
\Phi \longrightarrow \Phi(2pu\tau, p(-2\theta+r)-(s-1)u-2p\nu-u(s'-1)\tau-p(r'-2)\tau) \tag{4.2.56}
$$

We should now use the first identity in (3.4.7) by setting,

$$
\begin{aligned}
m &= p(-2\theta+r)-(s-1)u-1\ , \\
\mu &\to 1-2p\nu-u(s'-1)\tau-p(r'-2)\tau\ ,
\end{aligned}
$$

and simplify further the resulting Φ function by using the identity (3.4.10). We obtain,

$$
\begin{aligned}
&\Phi(2pu\tau, p(-2\theta+r)-(s-1)u-2p\nu-u(s'-1)\tau-p(r'-2)\tau) = \\
&-\Big[e^{-\frac{i\pi}{2pu\tau}[p(-2\theta+r)-(s-1)u-1]^2-\frac{i\pi}{pu\tau}[p(-2\theta+r)-(s-1)u-\frac{1}{2}]+\frac{i\pi}{pu\tau}[p(-2\theta+r)-(s-1)u][2p\nu+u(s'-1)\tau+p(r'-2)\tau]}\times \\
&\qquad\qquad \times\Phi(2pu\tau, 2p\nu+u(s'-2p-1)\tau+p(r'-2)\tau)\Big] \\
&\quad+\frac{i}{\sqrt{-2ipu\tau}}\sum_{j=1}^{p(-2\theta+r)-(s-1)u-1} e^{\frac{i\pi}{2pu\tau}[j^2-2pj(-2\theta+r)+2j(s-1)u]+\frac{i\pi j}{pu\tau}(2p\nu+u(s'-1)\tau+p(r'-2)\tau)}
\end{aligned}
\tag{4.2.57}
$$

Now putting the result above into (4.2.55) gives,

$$
\Delta\mathcal{H}_1 = \Delta\mathcal{H}_{11} + \Delta\mathcal{H}_{12}\ , \tag{4.2.58}
$$

where,

$$\Delta\mathcal{H}_{11} = \mathcal{M}'_{r,s,\theta}(\tau,\nu) \sum_{s'=1}^{2p} \sum_{r'=1}^{u} \sum_{\substack{n\in\mathbb{Z} \\ 1\le r'+2n\le 2u}} \Phi(2pu\tau, 2p\nu + u(s'-2p-1)\tau + p(r'-2)\tau)$$
$$\times\, e^{-2i\pi\frac{p}{u}[rn+\theta(r'-2)]+i\pi\frac{\tau}{2pu}(u(s'-1)-p(r'+2n-1))^2}$$
$$\times\, \Lambda_{s'-1,r'+2n,u,p}(\tau,0)\ , \tag{4.2.59}$$

and,

$$\Delta\mathcal{H}_{12} = \frac{\tau}{u} e^{2i\pi\frac{up}{\tau}(\frac{-r}{2u}+\frac{s-1}{2p})^2} \frac{i}{\sqrt{-2ipu\tau}} \sum_{j=1}^{p(-2\theta+r)-(s-1)u-1} e^{\frac{i\pi}{2pu\tau}[j^2-2pj(-2\theta+r)+2j(s-1)u]+\frac{2i\pi j\nu}{u\tau}}$$
$$\times \sum_{s'=1}^{2p} \sum_{r'=1}^{u} \sum_{\substack{n\in\mathbb{Z} \\ 1\le r'+2n\le 2u}} e^{i\pi(r+\frac{(s-1)u}{p})[s'-1-\frac{p}{u}(r'+2n-1)]+\frac{i\pi j}{up}[u(s'-1)+p(r'-2)]+i\pi\frac{u\tau}{2p}[s'-1-\frac{p}{u}(r'+2n-1)]^2}$$
$$\times\, \Lambda_{s'-1,r'+2n,u,p}(\tau,0) \tag{4.2.60}$$

Now to deal with $\Delta\mathcal{H}_{11}$, two points have to be argued. Firstly we should mention that to write (4.2.59) we have again used the equality (D.2.8). Secondly we have used the symbol $\mathcal{M}'_{r,s,\theta}(\tau,\nu)$ to represent all the possible factors in front of our triple sum, however in what follows it again changes to a yet another factor, namely $\mathcal{M}_{r,s,\theta}(\tau,\nu)$ (in terms of $\mathcal{C}_{-r,s,\theta}(\tau,\nu)$ and factors arising from (4.2.57), etc) which will shortly be introduced. Therefore to accomplish the term $\Delta\mathcal{H}_{11}$ as the only corrective term containing the Φ function, we first make the change $s' \to 2p - s' + 1$ and then use (C.2.4) and write,

$$\boxed{\begin{aligned}\Delta\mathcal{H}_{11} = {} & \mathcal{M}_{r,s,\theta}(\tau,\nu) e^{2i\pi\frac{p\nu^2}{u\tau}-i\pi\frac{p}{u}\frac{\tau}{2}} \sum_{s'=1}^{2p} \sum_{r'=1}^{u} \sum_{\substack{n\in\mathbb{Z} \\ 1\le r'+2n\le 2u}} \Phi(2pu\tau, 2p\nu - s'u\tau + p(r'-2)\tau) \\ & \times\, e^{-i\pi\frac{p}{u}[r(2n+1)+2\theta(r'-2)]+i\pi\frac{u\tau}{2p}(s'-\frac{p}{u}(r'+2n-1))^2} \\ & \times\, \Lambda_{s',r'+2n,u,p}(\tau,0)\ ,\end{aligned}} \tag{4.2.61}$$

where the final overall factor $\mathcal{M}_{r,s,\theta}(\tau,\nu)$, now is given by,

$$\mathcal{M}_{r,s,\theta}(\tau,\nu) = \frac{\tau}{u} e^{i\pi(s-1)+2i\pi\frac{up}{\tau}(\frac{-r}{2u}+\frac{s-1}{2p})^2-2i\pi\frac{p\nu^2}{u\tau}+i\pi\frac{p}{u}\frac{\tau}{2}} \times$$
$$\times\, e^{-\frac{i\pi}{2pu\tau}[p(-2\theta+r)-(s-1)u-1]^2-\frac{i\pi}{pu\tau}[p(-2\theta+r)-(s-1)u-\frac{1}{2}]+\frac{2i\pi\nu}{u\tau}[p(-2\theta+r)-(s-1)u]} \tag{4.2.62}$$

The term $\Delta\mathcal{H}_{12}$ in (4.2.60) can be further simplified as well. Implementing (D.2.6) along with further rearrangement of the sum indices makes it of a better shape to be compared with other corrective terms of our calculations. However we prefer to show this process in a same computational way, will be done for an equivalent term in the next chapter (called $\Delta\mathcal{R}_{12}$ in (5.2.31)).

• We intend now to treat $\mathcal{H}_2^-$ from (4.2.50) in a very similar way to $\mathcal{H}_1^-$. Therefore we try to express it using the suggested format in (D.2.9). First a quick look at $\mathcal{H}_2^-$ shows that all the functions in the summand are written either in the form $\Psi(r')$ or $f(r'+2n)$ except for $e^{-2i\pi n\tau(s'-\frac{p}{u}(r'+n))}$ which can be rearranged as $e^{-i\pi[(r'+2n)-r'][s'-\frac{p}{2u}(r'+2n)-\frac{pr'}{2u}]\tau}$. Now inserting this in $\mathcal{H}_2^-$ and using (D.2.9) we write,

$$\mathcal{H}_2^- = -\frac{\tau}{u}\mathcal{C}_{-r,s,\theta}(\tau,\nu)\sum_{s'=1}^{p-1}\sum_{\substack{r'=1\\ s'u-r'p>0}}^{u-1}\sum_{b=1-u}^{0} e^{2i\pi\eta^-(s'-\frac{p}{u}(2b-[r']_2))+2i\pi\gamma(s'-\frac{p}{u}r')}$$
$$\times\, e^{-i\pi[(2b-[r']_2)-r'][s'-\frac{p}{2u}(2b-[r']_2)-\frac{pr'}{2u}]\tau}\vartheta(2pu\tau, 2p\eta^- + s'u\tau - p(2b-[r']_2)\tau). \quad (4.2.63)$$

In analogy with $\mathcal{H}_1^\pm$ terms, it is not difficult now, to produce the $\mathcal{H}_2^+$ from the equality above. To avoid writing the long calculations we just mentioned, we again flip the sign of all $(2b-[r']_2)$ occurrences, using (D.2.11) (or using the knowledge about change of signs in appendix D.1), recall the value of γ from (4.2.25) and change $\eta^- \to \eta^+$ all through the formula above, subtract the resulting $\mathcal{H}_2^+$ from $\mathcal{H}_2^-$, rearrange the remaining sum indices and the final use of (D.2.8) to bring the situation back to the form of $\Psi(r')f(r'+2n)$, followed by using (C.2.4)(for $n=1$) and (D.2.6), we write the final result as [1],

$$\Delta\mathcal{H}_2 = \mathcal{H}_2^- - \mathcal{H}_2^+ = \mathcal{M}_{r,s,\theta}(\tau,\nu)\sum_{r'=1}^{2u}\sum_{s'=1}^{p-1}\sum_{\substack{b\in\mathbb{Z}\\ 2b+r'\geq 2\\ p(2b+r'-1)<s'u}} e^{-i\pi\frac{p}{u}[r(r'-1)+(-2\theta+r)(r'+2b)]-2i\pi\frac{p}{u}b^2\tau}$$
$$\times\, e^{i\pi(2b+1)(s'-\frac{p}{u}r')\tau-2i\pi\nu(s'-\frac{p}{u}(r'+2b))}\Lambda_{s',r',u,p}(\tau,0)\,. \quad (4.2.64)$$

• Now looking at the last corrective terms, we realise that there could be a chance

[1] Same as many calculations before, to avoid any mistake, all the mentioned steps are double checked by Maple. Meanwhile the most concise analytical way is also presented in [29].

to reform $\mathcal{G}_2$, one of the previous corrective terms shown in (4.2.13), to get unified with $\Delta\mathcal{H}_2$. In order to do that, we change $\mathcal{G}_2$ in an exponential style. Hence along with a change of $b \to b+1$ and $r' \to r'-1$ we write it as,

$$\mathcal{G}_2 = \frac{\tau}{u}\,\mathcal{C}_{-r,s,\theta}(\tau,\nu) \sum_{b=-u}^{u-1}\sum_{r'=1}^{u}\sum_{\substack{s'=1\\ s'u-r'p<p(2b+1)}}^{2p-1} e^{2i\pi[s'-\frac{p}{u}(r'+2b+1)]\nu} e^{2i\pi\frac{p}{u}[r(b+1)-\theta(r'+2b+1)]}$$
$$\times\; e^{i\pi\frac{p}{u}[r'+2b+1-2b(b+r')]\tau} e^{i\pi(2b+1)s'\tau}\Lambda_{s',r',u,p}(\tau,0)\;. \quad (4.2.65)$$

We again push the triple summation above by changing $s' \to 2p-s'$ and $b \to u-b-r'$. Then by using (C.2.4) and simplify the remaining prefactors, $\mathcal{G}_2$ reads,

$$\mathcal{G}_2 = -\mathcal{M}_{r,s,\theta}(\tau,\nu) \sum_{\substack{b=1-r'\\ p(2b+r'-1)<s'u}}^{2u-r'}\sum_{r'=1}^{u}\sum_{s'=1}^{2p-1} e^{-i\pi\frac{p}{u}[r(r'-1)+(-2\theta+r)(r'+2b)]-2i\pi\frac{p}{u}b^2\tau}$$
$$\times\; e^{i\pi(2b+1)(s'-\frac{p}{u}r')\tau-2i\pi\nu(s'-\frac{p}{u}(r'+2b))}\Lambda_{s',r',u,p}(\tau,0)\;. \quad (4.2.66)$$

As it can be seen, now the summand in formula above is exactly minus the one in (4.2.64). Therefore one should try to modify the indices for any possible unification. Here adding $\mathcal{G}_2$ and $\Delta\mathcal{H}_2$, we see that the triple sums combine as,

$$-\sum_{s'=1}^{2p-1}\sum_{r'=1}^{u}\sum_{\substack{b=1-r'\\ p(2b+r'-1)<s'u}}^{2u-r'} + \sum_{r'=1}^{2u}\sum_{b\in\mathbb{Z}}\sum_{\substack{s'=1\\ 2b+r'\geq 2\\ p(2b+r'-1)<s'u}}^{p-1} =$$
$$= \underbrace{-\sum_{s'=1}^{p-1}\sum_{r'=1}^{u}\sum_{\substack{b\geq 1-r'\\ p(2b+r'-1)<s'u}}}_{(1)} \underbrace{-\sum_{s'=p+1}^{2p-1}\sum_{r'=1}^{u}\sum_{\substack{b\geq 1-r'\\ p(2b+r'-1)<s'u}}}_{(1')} + \underbrace{\sum_{s'=1}^{p-1}\sum_{r'=1}^{u}\sum_{\substack{b\in\mathbb{Z}\\ 2b+r'\geq 2\\ p(2b+r'-1)<s'u}}}_{(2)} + \underbrace{\sum_{s'=1}^{p-1}\sum_{r'=u+1}^{2u}\sum_{\substack{b\in\mathbb{Z}\\ 2b+r'\geq 2\\ p(2b+r'-1)<s'u}}}_{(2')} \quad (4.2.67)$$

where in term (1) above we dropped the upper limit $s'=p$ using $\Lambda_{p,r',u,p=0}$ from (C.2.3). For both terms (1) and (1′) also, we have reformed the limits of index b since it never exceeds $2u-r'$. Therefore an essential replacement of summation indices in (1′) as $s' \to s'+p$, $r' \to r'-u$, $b \to b+u$ (by which, summands are quite noticeably preserved [2]) and recalling (C.2.5), we see that the terms 1′ and 2′

[2]We earlier realised this preservation, while using the Maple software to check some identities.

combine as,

$$(1') + (2') = -\sum_{s'=1}^{p-1} \sum_{r'=u+1}^{2u} \sum_{\substack{b \geq 1-r' \\ 2b \leq 1-r'}} \tag{4.2.68}$$

With (1) + (2), a similar triple sum appears straightforwardly, and therefore (after additionally changing the summation indices as $b \to b - r'$) we write,

$$\boxed{\begin{aligned}\mathcal{L}_2 = \mathcal{G}_2 + \Delta\mathcal{H}_2 = -\mathcal{M}_{r,s,\theta}(\tau,\nu) \sum_{b=1}^{u} \sum_{r'=2b-1}^{2u} \sum_{s'=1}^{p-1} e^{-i\pi\frac{p}{u}[r(r'-1)+(-2\theta+r)(r'+2b)]-2i\pi\frac{p}{u}b^2\tau} \\ \times e^{i\pi(2b+1)(s'-\frac{p}{u}r')\tau - 2i\pi\nu(s'-\frac{p}{u}(r'+2b))} \Lambda_{s',r',u,p}(\tau,0) \ . \quad (4.2.69)\end{aligned}}$$

We have now finished Step 2 of our calculation. However it is worthwhile to note the T-transform of $\varphi_{r,s,u,p}(\tau,\nu)$ using (3.3.1) (the even case) which happens very trivially as,

$$\varphi_{r,s,u,p;\theta}(\tau+1,\nu) = \varphi_{r,s,u,p;\theta}(\tau,\nu) \ , \tag{4.2.70}$$

makes the T-transformation of $N = 2$ characters a straightforward computation. We hence in the next subsection, concentrate only on S-transform of $N = 2$ characters.

4.2.2 S transform of $N = 2$ characters

We are now in a position to obtain the S-transform of the Ramond $N = 2$ admissible characters. Using (4.1.7), we may express the S-transform of $\omega_{r,s,u,p;\theta}(\tau,\nu)$ as,

$$\begin{aligned}&\omega_{r,s,u,p;\theta}(-\tfrac{1}{\tau},\tfrac{\nu}{\tau}) = \\ &= e^{\frac{2i\pi\nu}{\tau}[s-1-\frac{p}{u}(r-1-2\theta)]} e^{\frac{2i\pi}{\tau}[\frac{p}{u}\theta^2+(s-1-\frac{p}{u}r)\theta]} e^{-\frac{i\pi}{4\tau}} \frac{\vartheta_{1,0}(-\frac{1}{\tau},\frac{\nu}{\tau})}{\eta^3(-\frac{1}{\tau})} \varphi_{r,s,u,p;\theta}(-\tfrac{1}{\tau},\tfrac{\nu}{\tau}). \quad (4.2.71)\end{aligned}$$

Recalling (2.2.19), (2.2.24), (4.2.24) and

$$\vartheta_{1,0}(\tau,\tfrac{1}{2}+\nu-\tfrac{\tau}{2}) = -zq^{\frac{1}{2}}\vartheta_{1,0}(\tau,\tfrac{1}{2}+\nu+\tfrac{\tau}{2}), \tag{4.2.72}$$

we arrive at,

$$\omega_{r,s,u,p;\theta}(-\tfrac{1}{\tau},\tfrac{\nu}{\tau}) = -\tfrac{i}{\tau}\, zq^{\frac{1}{2}}\, e^{\frac{2i\pi\nu}{\tau}[s-1-\frac{p}{u}(r-1-2\theta)]}\, e^{\frac{i\pi}{\tau}(\nu^2-\nu)}$$
$$\times\, e^{\frac{2i\pi}{\tau}[\frac{p}{u}\theta^2+(s-1-\frac{p}{u}r)\theta]}\, \frac{\vartheta_{1,0}(\tau,\frac{1}{2}+\nu+\frac{\tau}{2})}{\eta^3(\tau)}\,[\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_c+\mathcal{L}_s+\Delta\mathcal{H}_{11}+\Delta\mathcal{H}_{12}]. \tag{4.2.73}$$

Note that the right-hand side of the above expression only contains $N=2$ characters within the term proportional to $\mathcal{L}_1$, since the latter is the only one proportional to functions φ (see (4.2.22)). All other terms are 'corrective terms' to the 'leading term' in the $N=2$ characters, and we treat them schematically here. We concentrate on $\mathcal{L}_1$ in what follows. Noting that,

$$\omega_{r',s,u,p;b-\frac{1}{2}}(\tau,\nu+\tfrac{1}{2}) = e^{\frac{i\pi}{2}}\, e^{i\pi(s-1-\frac{p}{u}[r'-2b])}\, z^{s-1-\frac{p}{u}(r'-2b)+\frac{1}{2}}\, q^{\frac{3}{8}-(b'-\frac{1}{2})[s-1+\frac{p}{u}[b-\frac{1}{2}-r']}$$
$$q^{\frac{1}{8}}\,\frac{\vartheta_{1,0}(\tau,\nu+\frac{1}{2}+\frac{\tau}{2})}{\eta^3(\tau)}\,\varphi_{r',s,u,p;b-\frac{1}{2}}(\tau,\nu+\tfrac{1}{2}), \tag{4.2.74}$$

we therefore present our final S transformation of non-unitary $N=2$ characters as,

$$\boxed{\begin{aligned}\omega_{r,s,u,p;\theta}(-\tfrac{1}{\tau},\tfrac{\nu}{\tau}) &= -\frac{1}{u}e^{\frac{i\pi}{\tau}(1-\frac{2p}{u})(\nu^2-\nu)}\, z^{\frac{1}{2}(1-\frac{2p}{u})}\\ &\times\sum_{r'=1}^{u-1}\sum_{b=1}^{u} e^{-\frac{i\pi p}{u}(rr'+1)}\, e^{\frac{i\pi p}{u}(2\theta-r-1)(2b-r'-1)}\,\omega_{r',s,u,p;b-\frac{1}{2}}(\tau,\nu+\tfrac{1}{2})\\ &-\frac{i}{\tau}\, zq^{\frac{1}{2}}\, e^{\frac{2i\pi\nu}{\tau}[s-1-\frac{p}{u}(r-1-2\theta)]}\, e^{\frac{i\pi}{\tau}(\nu^2-\nu)}\, e^{\frac{2i\pi}{\tau}[\frac{p}{u}\theta^2+(s-1-\frac{p}{u}r)\theta]}\\ &\times\frac{\vartheta_{1,0}(\tau,\frac{1}{2}+\nu+\frac{\tau}{2})}{\eta^3(\tau)}\,[\mathcal{L}_2+\mathcal{L}_c+\mathcal{L}_s+\Delta\mathcal{H}_{11}+\Delta\mathcal{H}_{12}]. \qquad (4.2.75)\end{aligned}}$$

In which all the corrective terms are respectively given by, (4.2.69), (4.2.23), (4.2.18), (4.2.59) and (4.2.60).

4.3 The very special case of unitary $N = 2$ superconformal characters

Minimal unitary $N = 2$ characters are given by (4.1.4) when $p = 1$. Although the parameter s can in principle take the values 0 and 1, one checks that,

$$\omega_{(r,1,u,1;\theta)}(q,z) = -\omega_{(r,0,u,1;\theta)}(q,z) \tag{4.3.1}$$

and we will always set $s = 1$ in the unitary $N = 2$ characters we discuss. Setting $p = s = 1$ in (4.2.75), and defining $\omega_{r,u;\theta}(q,z) \equiv \omega_{r,1,u,1;\theta}(q,z)$, one can show that all the corrective terms vanish (it is clear when using (C.2.3)) and we are left with the well-known transformation law [32],

$$\boxed{\begin{aligned}\omega_{r,u;\theta}\left(-\tfrac{1}{\tau},\tfrac{\nu}{\tau}\right) &= -\frac{1}{u}e^{\frac{i\pi}{\tau}\left(1-\frac{2}{u}\right)(\nu^2-\nu)}z^{\frac{1}{2}\left(1-\frac{2}{u}\right)} \\ &\quad\times\sum_{r'=1}^{u-1}\sum_{b'=1}^{u} e^{-\frac{i\pi}{u}(rr'+1)}\,e^{\frac{i\pi}{u}(2\theta-r-1)(2b'-r'-1)}\omega_{r',u;b'-\frac{1}{2}}\left(\tau,\nu+\tfrac{1}{2}\right).\end{aligned}} \tag{4.3.2}$$

In the case of unitary minimal characters however, the rewriting in terms of Appell functions is not crucial for the derivation of S modular transformations. This is due to the fact that, for unitary $N = 2$ characters, the functions (4.1.12) drastically simplify and yield an infinite product involving ratios of theta functions. By exploiting this property, we provide a derivation of S transforms which can be used as a consistency check of our formalism in the special case $p = 1$. One has,

$$\varphi_{(r,u;\theta)}(q,z) = \mathcal{K}_{(2)}(q^u, q^{\frac{r}{2}}, -z^{-1}q^{-\frac{r}{2}+\theta}) - \mathcal{K}_{(2)}(q^u, q^{-\frac{r}{2}}, -z^{-1}q^{-\frac{r}{2}+\theta}), \tag{4.3.3}$$

which actually is an infinite product: consider (2.4.24) for $p = 1$, namely

$$\mathcal{K}_{(2)}(q,x,y) - \mathcal{K}_{(2)}(q,x^{-1},y) = -\frac{\vartheta_{(1,1)}(q,x^2)\,q^{-\frac{1}{8}}\,\tilde{\eta}(q)^3}{\vartheta_{(1,1)}(q,xy)\vartheta_{(1,1)}(q,xy^{-1})}. \tag{4.3.4}$$

Now change $q \to q^u$, $x \to q^{\frac{r}{2}}$ and $y \to -z^{-1}q^{-\frac{r}{2}+\theta}$ in (4.3.4) to obtain (4.3.3) and hence the unitary $N = 2$ characters, as infinite products involving ratios of theta functions. We obtain [3].

$$\omega_{(r,u;\theta)}(q,z) = -q^{\frac{1-u}{8}} q^{-\frac{\theta}{u}(\theta-r)} z^{2\frac{\theta}{u}-\frac{r-1}{u}} \cdot \frac{\tilde{\eta}(q^u)^3}{\tilde{\eta}(q)^3} \cdot \frac{\vartheta_{(1,0)}(q,z)\ \vartheta_{(1,1)}(q^u,q^r)}{\vartheta_{(1,0)}(q^u,z^{-1}q^{\theta})\vartheta_{(1,0)}(q^u,zq^{r-\theta})}, \tag{4.3.5}$$

where we have used $\vartheta_{(1,1)}(q,-z)=\vartheta_{(1,0)}(q,z)$. The above product formula has been known for two decades [38], but its derivation did not involve the Appell functions at level 2 we have introduced in an effort to illustrate the general formula (4.1.12) in a simple case. We spend the rest of this section re-deriving the S modular transformation of unitary $N=2$ characters, taking full advantage of the simplification just described. **We emphasise once more that the approach followed below cannot be generalised easily to higher values of** p, as the non-unitary $N=2$ characters are not expressible as infinite products. Define the function,

$$F(\mu,\nu,\tau) = \frac{\prod_{n=1}^{\infty}(1-q^n)^3\vartheta_{1,1}(\nu-\mu,\tau)}{\vartheta_{1,0}(-\mu,\tau)\vartheta_{1,0}(\nu,\tau)}\ , \tag{4.3.6}$$

so that the unitary $N=2$ characters appearing in (4.3.5) are represented as,

$$\omega_{r,u;\theta}(\tau,\nu) = -e^{2i\pi\frac{2\theta-r+1}{u}\nu} e^{2i\pi\tau(\frac{-\theta^2+r\theta}{u}+\frac{1}{8})} \frac{\vartheta_{1,0}(\tau,\nu)}{\eta(\tau)^3} F(\nu-\theta\tau,\nu+(r-\theta)\tau,u\tau)\ . \tag{4.3.7}$$

So evaluating the S transform of these characters involves the following reformulation of (C.1.7),

$$F(\tfrac{\nu+\theta}{\tau},\tfrac{\nu+\theta-r}{\tau},\tfrac{-u}{\tau}) = \tfrac{\tau}{u} e^{-\frac{i\pi\tau}{2u}+2i\pi[\frac{\nu+\theta}{u}-\frac{\tau}{2u}+\frac{r(\nu+\theta)}{u\tau}-\frac{(\nu+\theta)^2}{u\tau}]} F(\tfrac{\nu+\theta}{u}-\tfrac{\tau}{2u}+\tfrac{1}{2},\tfrac{\nu+\theta-r}{u}-\tfrac{\tau}{2u}+\tfrac{1}{2},\tfrac{\tau}{u}) \tag{4.3.8}$$

and (2.2.19), (2.2.24). Using this we obtain,

$$\omega_{r,u;\theta}(\tfrac{-1}{\tau},\tfrac{\nu}{\tau}) = -\tfrac{1}{u} e^{2i\pi\frac{\nu+\theta}{u}+i\pi\frac{\nu^2-\nu}{u\tau}(u-2)-\frac{3i\pi}{2}-\frac{i\pi\tau}{2u}-\frac{i\pi r}{u}} \times$$
$$\times \frac{\vartheta_{1,0}(\tau,\nu-\frac{\tau}{2}+\frac{1}{2})}{\eta(\tau)^3} F(\tfrac{\nu+\theta}{u}-\tfrac{\tau}{2u}+\tfrac{1}{2},\tfrac{\nu+\theta-r}{u}-\tfrac{\tau}{2u}+\tfrac{1}{2},\tfrac{\tau}{u}). \tag{4.3.9}$$

This formula should be manipulated to be looking schematically like (2.2.28) and (2.2.29). To do that, we first introduce the following period-increasing formula for

[3]the presence of a factor $q^{\frac{1-u}{8}}$ is correlated to the definition we adopted for theta functions in (2.2.9)-(2.2.12)

the function F, proved in appendix (C.1),

$$F(\alpha, \beta, \tfrac{\tau}{u}) = \sum_{a,b=0}^{u-1} e^{2i\pi b\alpha - 2i\pi a\beta + i\pi(a+b) + 2i\pi \frac{ab\tau}{u}} F(u\alpha + \tau a + \tfrac{1+u}{2}, u\beta - \tau b + \tfrac{1+u}{2}, u\tau) \ . \tag{4.3.10}$$

Using the above, the F function in the right-hand side of (4.3.9) becomes,

$$\begin{aligned} &F(\tfrac{\nu+\theta}{u} - \tfrac{\tau}{2u} + \tfrac{1}{2}, \tfrac{\nu+\theta-r}{u} - \tfrac{\tau}{2u} + \tfrac{1}{2}, \tfrac{\tau}{u}) \\ &= \sum_{a,b=0}^{u-1} e^{2i\pi b\frac{\nu+\theta}{u} - 2i\pi a\frac{\nu+\theta-r}{u} + i\pi\frac{\tau}{u}(a-b) + 2i\pi\frac{ab\tau}{u}} F(\nu - \tfrac{\tau}{2} + \tau a + \tfrac{1}{2}, \nu - \tfrac{\tau}{2} - \tau b + \tfrac{1}{2}, u\tau), \end{aligned} \tag{4.3.11}$$

since

$$\begin{aligned} F(\nu + \tau(a - \frac{1}{2}) + \frac{1}{2} + \theta + u, \nu - \tau(b + \frac{1}{2}) + \frac{1}{2} + \theta + u - r, u\tau) = \\ = F(\nu + \tau(a - \frac{1}{2}) + \frac{1}{2}, \nu - \tau(b + \frac{1}{2}) + \frac{1}{2}, u\tau). \end{aligned} \tag{4.3.12}$$

We now have a double sum on period increased F functions. Rewriting these in terms of $N = 2$ characters using (4.3.7), and after some simplifications we obtain,

$$\begin{aligned} \omega_{r,u;\theta}(\tfrac{-1}{\tau}, \tfrac{\nu}{\tau}) = -\tfrac{1}{u} e^{2i\pi\frac{\theta}{u} + i\pi\frac{\nu^2-\nu}{u\tau}(u-2) - \frac{3i\pi}{2} - \frac{i\pi\tau}{4} + \frac{i\pi\tau}{2u} - \frac{i\pi(r+1)}{u}} \times \\ \times \sum_{a=0}^{u-1}\sum_{b=0}^{u-1} e^{\frac{2i\pi}{u}(b\theta - a\theta + ar + \frac{a}{2} - \frac{b}{2})} \omega_{-a-b,u;-a}(\tau, \nu + \tfrac{1}{2} - \tfrac{\tau}{2}) \ . \end{aligned} \tag{4.3.13}$$

We wish to make two remarks. Firstly it can be seen that the Ramond unitary minimal $N = 2$ characters after S transformation have become "super-NS" characters as we expected, but secondly, the first label of $N = 2$ characters in the linear combination in the r.h.s. is not in the fundamental range. In order to address this problem, we first show that the r.h.s of (4.3.11) may be written $\sum_{a,b=0}^{u-1} h(a, b)$ with,

$$h(a + u, b) = h(a, b + u) = h(a, b) \ , \tag{4.3.14}$$

and

$$h(a, u - a) = h(0, 0) = 0 \ . \tag{4.3.15}$$

We then write,

$$\begin{aligned}
\sum_{a=0}^{u-1}\sum_{b=0}^{u-1} h(a,b) &= \sum_{a=0}^{u-1}\sum_{b=0}^{u-a-1} h(a,b) + \sum_{a=0}^{u-1}\sum_{b=u-a}^{u-1} h(a,b) \\
&= \sum_{a=0}^{u-1}\sum_{b=a}^{u-1} h(a,b-a) + \sum_{a=0}^{u-1}\sum_{b=0}^{a-1} h(a,b+u-a) \\
= \sum_{a=0}^{u-1}\sum_{b=0}^{u-1} h(a,b-a) &- \sum_{a=0}^{u-1}\sum_{b=0}^{a-1} h(a,b-a) + \sum_{a=0}^{u-1}\sum_{b=0}^{a-1} h(a,b+u-a).
\end{aligned}$$

In the last line the second and the third sums cancel each other by using (4.3.14) and we are left with the first sum which by changing $a \to u-a$ and $b \to u-b$ turns out to be,

$$\sum_{a=1}^{u}\sum_{b=1}^{u} h(u-a,-b+a) = \sum_{a=1}^{u}\sum_{b=1}^{u} h(-a,a-b) = \sum_{a=1}^{u}\sum_{b=1}^{u-1} h(-a,a-b)\ , \tag{4.3.16}$$

in which we have used (4.3.15). By using the property above, with $a \to b'$ and $b \to r'$, (4.3.9) changes to,

$$\omega_{r,u;\theta}(\tfrac{-1}{\tau},\tfrac{\nu}{\tau}) = -\tfrac{1}{u}\, e^{i\pi(\frac{\nu^2-\nu}{\tau}+\nu)(1-\frac{2}{u})} \sum_{r'=1}^{u-1}\sum_{b'=1}^{u} e^{\frac{i\pi}{u}(2\theta-r+4\theta b'-2r'\theta-2rb'+r'-2b'-2)} \omega_{r',u;b'+\frac{1}{2}}(\tau,\nu+\tfrac{1}{2})\ , \tag{4.3.17}$$

where we have used (4.1.9). We change $b' \to b'-1$ and recall that $\omega_{r',u;b'-\frac{1}{2}+u}(\tau,\nu+\frac{1}{2}) = \omega_{r',u;b'-\frac{1}{2}}(\tau,\nu+\frac{1}{2})$, and we finally get (4.3.2). We therefore have performed a valuable consistency check on our general formula (4.2.75).

Chapter 5

Modular transforming the $\widehat{s\ell}(2|1)$ characters via Appell functions

5.1 Introduction and basic definitions

We now apply the 'Appell function method' to the derivation of the modular transformations of admissible $\widehat{s\ell}(2|1)$ characters. The corresponding representations have been the object of much study in the recent past as the affine Lie superalgebra $\widehat{s\ell}(2|1)$ at fractional level

$$k = \frac{p}{u} - 1, \qquad p \text{ and } u \text{ coprime}, \tag{5.1.1}$$

appears to be relevant to a particular description of $N = 2$ non-critical strings, which are the prototype of $N = 2$ supergravity in two dimensions [24, 35, 40–43].

The infinite-dimensional algebra $\widehat{s\ell}(2|1)$ is induced by the superalgebra $sl(2/1)$ which possesses four 'bosonic' or 'even' generators $J^{\pm}, J^3, U$ and four 'fermionic' or 'odd' generators $j^{\pm}, j^{\pm,\prime}$. To see this, one promotes the eight generators to being functions of a complex variable z: $J^{\pm}(z), J^3(z)$ and $U(z)$ generating the even affine subalgebra $\widehat{sl(2)} \times \widehat{u(1)}$, and the remaining currents being $j^{\pm}(z), j^{\pm\prime}(z)$. One then assumes that the eight currents satisfy periodic boundary conditions of the type,

$$\mathcal{J}(e^{2i\pi}z) = \mathcal{J}(z) \tag{5.1.2}$$

so that their Laurent expansion is given by,

$$\mathcal{J}(z) = \sum_{n\in\mathbb{Z}} \mathcal{J}_n z^{-n-1}. \tag{5.1.3}$$

The non vanishing (anti)commutation relations between Laurent modes are,

$$\begin{aligned}
[J^+_m, J^-_n] &= 2J^3_{m+n} + \tilde{k}m\delta_{m+n,0}, & [J^3_m, J^\pm_n] &= \pm J^\pm_{m+n}, \\
[J^\pm_m, j'^\mp_n] &= \pm j^\pm_{m+n}, & [J^\pm_m, j^\mp_n] &= \mp j'^\pm_{m+n}, \\
[2J^3_m, j'^\pm_n] &= \pm j'^\pm_{m+n}, & [2J^3_m, j^\pm_n] &= \pm j^\pm_{m+n}, \\
[2U_m, j'^\pm_n] &= \pm j'^\pm_{m+n}, & [2U_m, j^\pm_n] &= \mp j^\pm_{m+n}, \\
[J^3_m, J^3_n] &= \tfrac{\tilde{k}}{2}m\delta_{m+n,0}, & [U_m, U_n] &= -\tfrac{\tilde{k}}{2}m\delta_{m+n,0}, \\
[j'^+_m, j'^-_n]_+ &= U_{m+n} - J^3_{m+n} - m\tilde{k}\delta_{m+n,0} & &, \\
[j^+_m, j^-_n]_+ &= U_{m+n} + J^3_{m+n} + m\tilde{k}\delta_{m+n,0} \\
[j'^+_m, j^+_n]_+ &= J^+_{m+n}, & [j'^-_m, j^-_n]_+ &= J^-_{m+n},
\end{aligned} \tag{5.1.4}$$

and the zero-modes (anti)-commutators close among themselves to yield the commutation relations of $sl(2/1)$, whose Cartan subalgebra is generated by J^3_0 and U_0. The following quadratic expression in the currents

$$T(z) = \frac{1}{k+1}\left[J^3J^3 - UU + J^+J^- + j^{+'}j^{-'} - j^+j^-\right](z), \tag{5.1.5}$$

is the energy-momentum tensor of the theory and its Laurent modes generate a Virasoro algebra with zero central charge. This very particular value of central charge is related to the fact that $\widehat{s\ell}(2|1)$ has an equal number of even and odd generators. It is common in many physical applications to consider the semi-direct product of the affine superalgebra and the Virasoro algebra emerging from the energy-momentum tensor construction. The zero-mode of (5.1.5), L_0, spans the Cartan subalgebra together with $J^3_0, U_0, \tilde{k}$ where the central element $\tilde{k}$ has eigenvalue k (the level appearing in (5.1.4)).

An $\widehat{s\ell}(2|1)$ highest weight $|\Omega\rangle$ at level k is characterised by its isospin h_-, hypercharge h_+ and conformal weight Δ. We write $|\Omega\rangle = |h_-, h_+, k\rangle$ with

$$U_0|h_-, h_+, k\rangle = h_+|h_-, h_+, k\rangle, \qquad J^3_0|h_-, h_+, k\rangle = h_-|h_-, h_+, k\rangle \tag{5.1.6}$$

and

$$L_0|h_-,h_+,k\rangle = \Delta_{h_-,h_+,k}|h_-,h_+,k\rangle, \tag{5.1.7}$$

where the conformal weight $\Delta_{h_-,h_+,k}$ can be read off the expression (5.1.5) for the energy-momentum tensor. It is given by,

$$\Delta_{h_-,h_+,k} = \frac{h_-^2 - h_+^2}{k+1}. \tag{5.1.8}$$

The annihilation conditions for the highest weight state are, [37]

$$j'^+_0|h_-,h_+,k\rangle = 0, \qquad j_0^+|h_-,h_+,k\rangle = 0, \qquad J_1^-|h_-,h_+,k\rangle = 0. \tag{5.1.9}$$

A Verma module with highest weight state $|h_-,h_+,k\rangle$ is the set of all states obtained by the application of negative mode generators on that highest weight state, taking the relations (5.1.4) into account. The corresponding characters are formally given by a trace over the Verma module $V_{h_-,h_+,k}$,

$$\chi^V_{h_-,h_+,k}(q,x,y) = Tr_{V_{h_-,h_+,k}}(q^{L_0}x^{J_0^3}y^{U_0}), \tag{5.1.10}$$

where q, x and y are three complex variables, $q = e^{2i\pi\tau}$, $x = e^{2i\pi\nu}$ and $y = e^{2i\pi\mu}$, with $\tau,\nu,\mu \in \mathbb{C}$ and $Im(\tau) > 0$ for convergence purposes. Explicitly, one has [3],

$$\chi^V_{h_-,h_+,k}(q,x,y) = x^{h_-}y^{h_+}q^{\frac{h_-^2-h_+^2}{k+1}}\frac{\vartheta_{(1,0)}(q,x^{\frac{1}{2}}y^{\frac{1}{2}})\vartheta_{(1,0)}(q,x^{\frac{1}{2}}y^{-\frac{1}{2}})}{\vartheta_{(1,1)}(q,z)\prod_{i\geq 1}(1-q^i)^3}. \tag{5.1.11}$$

If h_- and h_+ satisfy certain conditions [1], the application of negative mode generators on the highest weight state $|h_-,h_+,k\rangle$ will produce states of zero norm annihilated by j'^+_0, j_0^+ and J_1^-: these are called singular vectors. Irreducible representations are obtained by removing from the initial Verma module the states of zero norm and their descendants. This procedure is not at all trivial and was carried out in [40]. The corresponding irreducible characters were first given in [24] and later rewritten in the form we use here by A. Semikhatov and A. Taormina. We consider the $\widehat{s\ell}(2|1)$ algebra at level $k = \frac{p}{u} - 1$ and the class of representations with

$$h_- = \frac{r-1}{2} - \frac{s-1}{2}\frac{p}{u}, \qquad 1-p \leq r \leq p, \qquad 1 \leq s \leq u, \tag{5.1.12}$$

[1] see [26] for a discussion

and

$$h_- - h_+ = \frac{p}{u}. \tag{5.1.13}$$

The character formula (5.1.11) acquires a 'corrective' factor $\psi_{(r,s,u,p)}(q,x,y)$ which takes into account the modding out by submodules with a singular vector as highest weight state. Explicitly, the $\widehat{s\ell}(2|1)$ admissible characters are given by,

$$\chi_{(r,s,u,p)}(q,x,y) = x^{\frac{r-1}{2}-\frac{s-1}{2}\frac{p}{u}} y^{\frac{r-1}{2}-\frac{s+1}{2}\frac{p}{u}} q^{r-1-s\frac{p}{u}} \times \frac{\vartheta_{(1,0)}(q,x^{\frac{1}{2}}y^{\frac{1}{2}})\vartheta_{(1,0)}(q,x^{\frac{1}{2}}y^{-\frac{1}{2}})}{\vartheta_{(1,1)}(q,x)\prod_{i\geq 1}(1-q^i)^3}\psi_{(r,s,u,p)}(q,x,y), \tag{5.1.14}$$

with,

$$\psi_{(r,s,u,p)}(q,x,y) = \sum_{m\in\mathbb{Z}} q^{m^2up-mu(r-1)}\left(\frac{q^{mp(s-1)}x^{-mp}}{1+x^{-\frac{1}{2}}y^{-\frac{1}{2}}q^{mu-1}} - q^{(s-1)(r-1)}x^{1-r}\frac{q^{-mp(s-1)}x^{mp}}{1+x^{\frac{1}{2}}y^{-\frac{1}{2}}q^{mu-s}}\right). \tag{5.1.15}$$

The integer-twisted characters are labelled by integers r, s, θ such that

$$1-p \leq r \leq p, \qquad 1 \leq s \leq u, \qquad \theta \in \mathbb{Z}. \tag{5.1.16}$$

They are given by,

$$\chi_{(r,s,u,p;\theta)}(q,x,y) = y^{-k\theta}q^{-k\theta^2}\chi_{(r,s,u,p)}(q,x,yq^{2\theta}). \tag{5.1.17}$$

As for admissible $N=2$ characters, the difficulty in deriving the modular properties of the above characters lies in the non-quasiperiodicity of the spectral flow parameter θ in the general case where $p \neq 1$ in (5.1.1). Also, to obtain Neveu-Schwarz characters from Ramond characters (see Chapter 4 for a detailed discussion of this point), one must allow for spectral flow parameters of the form $\theta \pm \frac{1}{2}, \theta \in \mathbb{Z}$. We have,

$$\begin{aligned}\chi_{(r,s,u,p;\theta\pm\frac{1}{2})}(q,x,y) &= y^{-k(\theta\pm\frac{1}{2})}q^{-k(\theta\pm\frac{1}{2})^2}\chi_{(r,s,u,p)}(q,x,yq^{2\theta\pm 1}) \\ &= y^{\mp\frac{k}{2}}q^{-\frac{k}{4}}\chi_{(r,s,u,p;\theta)}(q,x,yq^{\pm 1}).\end{aligned} \tag{5.1.18}$$

We are now ready to discuss the behaviour of the admissible characters (5.1.17) under the modular group.

5.2 Transformations of non-unitary $\widehat{s\ell}(2|1)$ characters

We start by rewriting $\psi_{r,s,u,p;\theta}(q,x,y) \equiv \psi_{(r,s,u,p)}(q,x,yq^{2\theta})$ in a way that will make the S-transform of the full $\widehat{s\ell}(2|1)$ characters easier to derive. Using (5.1.15), we get

$$\psi_{(r,s,u,p;\theta)}(q,x,y) = \sum_{m\in\mathbb{Z}} q^{m^2up+mu(r-1)}\Big(\frac{q^{-mp(s-1)+mu+1+\theta}x^{mp}x^{\frac{1}{2}}y^{\frac{1}{2}}}{1+x^{\frac{1}{2}}y^{\frac{1}{2}}q^{mu+1+\theta}} - q^{(s-1)(r-1)}x^{1-r}\frac{q^{mp(s-1)+mu+s+\theta}x^{-mp}x^{-\frac{1}{2}}y^{\frac{1}{2}}}{1+x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{mu+s+\theta}}\Big). \quad (5.2.1)$$

Now we re-express $\psi_{(r,s,u,p;\theta)}(q,x,y)$ in terms of $\psi_{(1,s,u,p;\theta)}(q,x,y)$ at the cost of introducing the function

$$\Gamma_{r,s,u,p;\theta}(q,x,y) = \sum_{a=0}^{r-1}(-x^{-\frac{1}{2}}y^{-\frac{1}{2}}q^{-\theta-1})^{-a}\Lambda_{(a,s,u,p)}(q,x)\ . \quad (5.2.2)$$

We have

$$\psi_{(r,s,u,p;\theta)}(q,x,y) = \\ = (-x^{-\frac{1}{2}}y^{-\frac{1}{2}}q^{-\theta-1})^{r-1}\Big(\psi_{(1,s,u,p;\theta)}(q,x,y) + \Gamma_{r,s,u,p;\theta}(q,x,y)\Big). \quad (5.2.3)$$

To prove the very helpful reduction (5.2.3), we recall the definition of the Λ function given in (C.2.1) and write the Γ function as,

$$\Gamma_{r,s,u,p;\theta}(q,x,y) = \sum_{a=0}^{r-1}\Big[(-x^{\frac{1}{2}}y^{\frac{1}{2}}q^{1+\theta})^{a}\theta(q^{2up},x^{p}q^{au-p(s-1)}) \\ - (-x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{s+\theta})^{a}\theta(q^{2up},x^{p}q^{-au-p(s-1)})\Big]\ ,$$

or again, using (2.2.11),

$$\Gamma_{r,s,u,p;\theta}(q,x,y) = \sum_{m\in\mathbb{Z}} q^{m^2up-mp(s-1)}x^{mp}\Big[\sum_{a=0}^{r-1}(-1)^{a}x^{\frac{a}{2}}y^{\frac{a}{2}}q^{a(mu+1+\theta)}) \\ - \sum_{a=0}^{r-1}(-1)^{a}x^{-\frac{a}{2}}y^{\frac{a}{2}}q^{a(-mu+s+\theta)})\Big]\ . \quad (5.2.4)$$

Now by taking $A = -x^{\frac{1}{2}}y^{\frac{1}{2}}q^{mu+1+\theta}$ and using the lemma (2.4.14) one obtains,

$$\sum_{a=0}^{r-1}A^{a} = \frac{1-(-x^{\frac{1}{2}}y^{\frac{1}{2}}q^{mu+1+\theta})^{r}}{1+x^{\frac{1}{2}}y^{\frac{1}{2}}q^{mu+1+\theta}}\ , \quad (5.2.5)$$

and taking $B = x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{-mu+s+\theta}$ similarly gives,

$$\sum_{a=0}^{r-1} B^a = \frac{1-(-x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{-mu+s+\theta})^r}{1+x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{-mu+s+\theta}} . \tag{5.2.6}$$

Therefore inserting (5.2.5) and (5.2.6) in (5.2.4) gives,

$$\Gamma_{r,s,u,p;\theta}(q,x,y) = \sum_{m\in\mathbb{Z}} q^{m^2up-mp(s-1)}x^{mp}\Big[\frac{1-e^{i\pi r}x^{\frac{r}{2}}y^{\frac{r}{2}}q^{r(mu+1+\theta)}}{1+x^{\frac{1}{2}}y^{\frac{1}{2}}q^{mu+1+\theta}} - \frac{1-e^{i\pi r}x^{-\frac{r}{2}}y^{\frac{r}{2}}q^{r(-mu+s+\theta)}}{1+x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{-mu+s+\theta}}\Big]. \tag{5.2.7}$$

Now we write $\psi_{1,s,u,p;\theta}(q,x,y)$ by putting $r=1$ in (5.2.1) as,

$$\psi_{(1,s,u,p;\theta)}(q,x,y) = \sum_{m\in\mathbb{Z}} q^{m^2up-mp(s-1)}x^{mp}\Big[\frac{x^{\frac{1}{2}}y^{\frac{1}{2}}q^{mu+1+\theta}}{1+x^{\frac{1}{2}}y^{\frac{1}{2}}q^{mu+1+\theta}} - \frac{x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{-mu+s+\theta}}{1+x^{-\frac{1}{2}}y^{\frac{1}{2}}q^{-mu+s+\theta}}\Big]. \tag{5.2.8}$$

Adding together formulas (5.2.7) and (5.2.8) and multiplying the sum by the factor $(-x^{-\frac{1}{2}}y^{-\frac{1}{2}}q^{-\theta-1})^{r-1}$, we easily obtain (5.2.1) and we therefore have proved (5.2.3). In view of the formula (5.1.14), the most non trivial part of non-unitary $\widehat{s\ell}(2|1)$ characters to S-transform is $\psi_{r,s,u,p;\theta}$. Thanks to (5.2.3), it amounts to S-transform $\Gamma_{r,s,u,p;\theta}$ and $\psi_{(1,s,u,p;\theta)}$.

Step 1: S-transform of $\Gamma_{r,s,u,p;\theta}$

Using (C.2.2) we write,

$$\Lambda_{a,s,u,p}(\tfrac{-1}{\tau},\tfrac{1}{\nu}) = \vartheta(\tfrac{-2pu}{\tau},\tfrac{p\nu}{\tau}+\tfrac{p(s-1)}{\tau}-\tfrac{au}{\tau}) - e^{-2i\pi(\nu+s-1)\frac{a}{\tau}}\vartheta(\tfrac{-2pu}{\tau},\tfrac{p\nu}{\tau}+\tfrac{p(s-1)}{\tau}+\tfrac{au}{\tau}). \tag{5.2.9}$$

The only difference between the two theta functions in the formula above is the change of sign of $\frac{au}{\tau}$ in the second argument. This gives us the ability to compute the S-transform of both theta functions using (2.2.20), i.e.

$$\begin{aligned}
\vartheta(\tfrac{-2pu}{\tau},\tfrac{p\nu}{\tau}+\tfrac{p(s-1)}{\tau}\pm\tfrac{au}{\tau}) &= \vartheta(\tfrac{1}{\tau/2pu},\tfrac{(p\nu+p(s-1)\pm au)/2pu}{\tau/2pu})\\
&= \sqrt{\tfrac{-i\tau}{2pu}}e^{\frac{(p\nu+p(s-1)\pm au)^2}{2pu\tau}}\vartheta(\tfrac{\tau}{2pu},\tfrac{p\nu+p(s-1)\pm au}{2pu})\\
&= \sqrt{\tfrac{-i\tau}{2pu}}e^{\frac{i\pi}{2pu\tau}(p^2(\nu+s-1)^2+a^2u^2)}e^{\pm\frac{i\pi a}{\tau}(\nu+s-1)}\vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}\pm\tfrac{a}{2p}) .
\end{aligned}$$

We may therefore rewrite (5.2.9) as,

$$\Lambda_{a,s,u,p}(\tfrac{-1}{\tau},\tfrac{1}{\nu}) = \sqrt{\tfrac{-i\tau}{2pu}}\, e^{\frac{i\pi}{2pu\tau}(p^2(\nu+s-1)^2+a^2u^2)} e^{-\frac{i\pi a}{\tau}(\nu+s-1)} \times$$
$$\times \left[\vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}-\tfrac{a}{2p}) - \vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}+\tfrac{a}{2p})\right] . \quad (5.2.10)$$

To re-express the right-hand side of (5.2.10) in terms of Λ [2], we use (D.1.3) to write the first theta function above as,

$$\vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}-\tfrac{a}{2p}) = \sum_{r'=1}^{2p}\sum_{s'=1}^{u} e^{2i\pi(\frac{\nu+s-1}{2u}-\frac{a}{2p})[u(r'-1)-p(s'-1)]+i\pi\frac{\tau}{2pu}[u(r'-1)-p(s'-1)]^2}$$
$$\times\, \vartheta(2pu\tau, p\nu + u(r'-1)\tau - p(s'-1)\tau). \quad (5.2.11)$$

and the second one as,

$$\vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}+\tfrac{a}{2p}) = \sum_{r'=1}^{2p}\sum_{s'=1}^{u} e^{2i\pi(\frac{\nu+s-1}{2u}+\frac{a}{2p})[-u(r'-1)-p(s'-1)]+i\pi\frac{\tau}{2pu}[-u(r'-1)-p(s'-1)]^2}$$
$$\times\, \vartheta(2pu\tau, p\nu - u(r'-1)\tau - p(s'-1)\tau). \quad (5.2.12)$$

Thus the subtraction of (5.2.12) from (5.2.11) readily gives,

$$\left[\vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}-\tfrac{a}{2p}) - \vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}+\tfrac{a}{2p})\right] =$$
$$\sum_{r'=1}^{2p}\sum_{s'=1}^{u} e^{2i\pi(\frac{\nu+s-1}{2u}-\frac{a}{2p})[u(r'-1)-p(s'-1)]+i\pi\frac{\tau}{2pu}[u(r'-1)-p(s'-1)]^2}\Lambda_{r'-1,s',u,p}(\tau,\nu)$$
$$\equiv \sum_{r'=1}^{2p}\sum_{s'=1}^{u} f(r'-1,s'-1)\Lambda_{r'-1,s',u,p}(\tau,\nu)$$
$$= \sum_{r'=0}^{2p-1}\sum_{s'=1}^{u} f(r',s'-1)\Lambda_{r',s',u,p}(\tau,\nu)$$
$$= \sum_{r'=1}^{2p}\sum_{s'=1}^{u} f(r',s'-1)\Lambda_{r',s',u,p}(\tau,\nu)\ , \quad (5.2.13)$$

where the (C.2.3) has been used. Inserting the above in (5.2.10), we finally end up

[2]Our philosophy throughout our thesis is to re-express the S-transforms in terms of Λ-type functions as the violating terms of unitary cases, rather than single ϑ functions

with,

$$\Lambda_{a,s,u,p}(\tfrac{-1}{\tau},\tfrac{1}{\nu}) = \sqrt{\tfrac{-i\tau}{2pu}}\, e^{\frac{i\pi}{2pu\tau}(p^2(\nu+s-1)^2+a^2u^2)} e^{-\frac{i\pi a}{\tau}(\nu+s-1)} \times$$
$$\times \sum_{r'=1}^{2p}\sum_{s'=1}^{u} e^{2i\pi(\frac{\nu+s-1}{2u}-\frac{a}{2p})[ur'-p(s'-1)]+i\pi\frac{\tau}{2pu}[ur'-p(s'-1)]^2}\Lambda_{r',s',u,p}(\tau,\nu)\ . \quad (5.2.14)$$

Using this outcome, the S-transform of the Γ function in (5.2.2) is given by,

$$\Gamma_{r,s,u,p}(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) = \sqrt{\tfrac{-i\tau}{2pu}}\sum_{a=0}^{r-1}\sum_{r'=1}^{2p}\sum_{s'=1}^{u} e^{i\pi a(1+\frac{\nu}{\tau}+\frac{\mu}{\tau}-(\theta+1)\frac{2}{\tau})}$$
$$\times\, e^{i\pi\frac{pu}{2\tau}(\frac{\nu+s-1}{u}-\frac{a}{p})^2+i\pi u(\frac{\nu+s-1}{u}-\frac{a}{p})[r'-\frac{p}{u}(s'-1)]+i\pi\frac{u\tau}{2p}[r'-\frac{p}{u}(s'-1)]^2}\Lambda_{r',s',u,p}(\tau,\nu)\ . \quad (5.2.15)$$

Step 2: S**-transform of** $\psi_{1,s,u,p;\theta}(q,x,y)$

We rewrite $\psi_{1,s,u,p;\theta}(q,x,y)$ in terms of Appell functions (2.3.5) as,

$$\psi_{1,s,u,p}(\tau,\nu,\mu) = \mathcal{K}_{2p}(u\tau,-\tfrac{\nu}{2}+\tfrac{s-1}{2}\tau,\tfrac{1}{2}-\tfrac{\mu}{2}-\tau\tfrac{s+1}{2})$$
$$-\,\mathcal{K}_{2p}(u\tau,\tfrac{\nu}{2}-\tfrac{s-1}{2}\tau,\tfrac{1}{2}-\tfrac{\mu}{2}-\tau\tfrac{s+1}{2}). \quad (5.2.16)$$

Hence using the key formula (3.4.21) and in very close analogy with all the calculations from (4.2.1) to (4.2.6) we come up with,

$$\psi_{1,s,u,p;\theta}(-\tfrac{1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) = \mathcal{Q}+\mathcal{R}\ , \quad (5.2.17)$$

where,

$$\mathcal{Q} = \tfrac{\tau}{u}\, e^{i\pi p\frac{\nu^2-\mu^2}{2u\tau}-2i\pi\frac{p}{u\tau}(\theta+s)(\theta+1)+i\pi\frac{p}{u\tau}(s-1)\nu+i\pi\frac{p}{u}(\frac{\mu}{\tau}-1)(2\theta+s+1)+i\pi\frac{p}{u}(\mu-\frac{\tau}{2})}$$
$$\times\left[\mathcal{K}_{2p}(\tfrac{\tau}{u},-\tfrac{\nu+s-1}{2u},\tfrac{\tau-\mu+2\theta+s+1}{2u})-\mathcal{K}_{2p}(\tfrac{\tau}{u},\tfrac{\nu+s-1}{2u},\tfrac{\tau-\mu+2\theta+s+1}{2u})\right], \quad (5.2.18)$$

and,

$$\mathcal{R} = \tfrac{\tau}{2pu}\, e^{i\pi\frac{p}{2u\tau}(\nu+s-1)^2}$$
$$\times\sum_{a=0}^{2p-1} e^{i\pi\frac{a^2u}{2p\tau}+i\pi\frac{a}{\tau}(\tau-\mu+2\theta+s+1)}\Phi(\tfrac{\tau}{2pu},\tfrac{\tau-\mu+2\theta+s+1}{2u}+\tfrac{a}{2p})$$
$$\times\left[\vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}+\tfrac{a}{2p})-\vartheta(\tfrac{\tau}{2pu},\tfrac{\nu+s-1}{2u}-\tfrac{a}{2p})\right]. \quad (5.2.19)$$

Again to relate the right hand side of (5.2.18) to the non-unitary characters we must increase the periods, and it can not be done without the help of the remarkable result in (2.4.21). As we already know, this result does not only provide the possibility of having a proper leading term for the S-transformed $\widehat{s\ell}(2|1)$ characters, but it also yields an interpretation in terms of some useful Λ functions. Setting $q \to q^{\frac{1}{u}}$, $s \to s'$ and taking $x = e^{-2i\pi\frac{\nu+s-1}{2u}}$ and $y = e^{2i\pi\frac{\tau-\mu+2\theta+s+1}{2u}}$ in (2.4.21) and using (2.4.6), we find, after tedious manipulations,

$$\mathcal{Q} = \mathcal{N}_{s,\theta}(\tau,\nu,\mu)\Bigg[\sum_{b=0}^{u-1}\sum_{s'=1}^{u} e^{-i\pi\frac{p}{u}(s-1)(s'-1)+i\pi\frac{p}{u}(2\theta+s+1)(2b+s')-2i\pi\frac{p}{u}b^2\tau} \\ \times e^{-i\pi\frac{p}{u}(2b+1)s'\tau-i\pi\frac{p}{u}(2b+s')\mu-i\pi\frac{p}{u}(s'-1)\nu}\psi_{1,s',u,p;b}(\tau,\nu,\mu-1-\tau)+\mathcal{Q}'\Bigg], \quad (5.2.20)$$

where

$$\mathcal{Q}' = \sum_{\substack{b=-u \\ ur'-ps'<p(2b+1)}}^{u-1}\sum_{r'=1}^{2p-1}\sum_{s'=1}^{u} e^{-i\pi\frac{p}{u}(s-1)(s'-1)+i\pi\frac{p}{u}(2\theta+s+1)(2b+s')-2i\pi\frac{p}{u}b^2\tau} \\ \times e^{i\pi(2b+1)(r'-\frac{p}{u}s')\tau+i\pi(r'-\frac{p}{u}(2b+s'))\mu+i\pi(r'-\frac{p}{u}(s'-1))\nu}\Lambda_{r',s',u,p}(\tau,\nu)\ , \quad (5.2.21)$$

and where we have isolated the overall factor,

$$\mathcal{N}_{s,\theta}(\tau,\nu,\mu) = \tfrac{\tau}{u}\, e^{i\pi p\frac{\nu^2-\mu^2}{2u\tau}-2i\pi\frac{p}{u\tau}(\theta+s)(\theta+1)+i\pi\frac{p}{u\tau}(s-1)\nu+i\pi\frac{p}{u\tau}(2\theta+s+1)\mu+i\pi\frac{p}{u}\frac{\tau}{2}}. \quad (5.2.22)$$

It serves our purpose to rewrite $\mathcal{Q}'$ by changing $r' \to 2p-r'$ and $b \to u-b-s'$ as well as using the shift-reflecting identity in (C.2.4) for $n=1$. We get,

$$\mathcal{Q}' = -\sum_{\substack{r'=1 \\ p(2b+s'-1)<ur'}}^{2p-1}\sum_{s'=1}^{u}\sum_{b=1-s'}^{2u-s'} e^{-i\pi\frac{p}{u}(s-1)(s'-1)-i\pi\frac{p}{u}(2\theta+s+1)(2b+s')-2i\pi\frac{p}{u}b^2\tau} \\ \times e^{i\pi(2b+1)(r'-\frac{p}{u}s')\tau-i\pi(r'-\frac{p}{u}(2b+s'))\mu+i\pi(r'-\frac{p}{u}(s'-1))\nu}\Lambda_{r',s',u,p}(\tau,\nu)\ . \quad (5.2.23)$$

Now having obtained the *leading term* in (5.2.20) and the associated corrective term $\mathcal{Q}'$ in (5.2.23), we need to deal with the $\mathcal{R}$ term appearing in (5.2.17). In order to proceed, we use (C.3.6), in which in accordance with (5.2.19) we change,

$$\begin{aligned} r' &\leftrightarrow s' \\ \gamma &\to \tfrac{1}{2}(\tau-\mu+2\theta+s+1) \\ \eta &\to -\tfrac{\nu+s-1}{2}\ , \end{aligned} \quad (5.2.24)$$

and write,

$$\mathcal{R} = (\mathcal{R}_1^- - \mathcal{R}_1^+) + (\mathcal{R}_2^- - \mathcal{R}_2^+) \ , \tag{5.2.25}$$

in which we define again,

$$\begin{aligned}
\mathcal{R}_1^- =& \frac{\tau}{u}\, e^{i\pi \frac{p}{2u\tau}(\nu+s-1)^2} \\
&\times \sum_{r'=1}^{2p} \sum_{s'=1}^{u} \sum_{\substack{n\in\mathbb{Z} \\ 1\le s'+2n\le 2u}} e^{i\pi(\nu+s-1)((r'-1)-\frac{p}{u}(s'+2n-1))+i\pi\frac{\tau}{2pu}(u(r'-1)-p(s'+2n-1))^2} \\
&\qquad \times \Phi(2pu\tau, p(\tau-\mu+2\theta+s+1)-u(r'-1)\tau-p(s'-1)\tau) \\
&\qquad \times \vartheta(2pu\tau, p\nu+u(r'-1)\tau+p(s'+2n-1)\tau) \ ,
\end{aligned} \tag{5.2.26}$$

and,

$$\begin{aligned}
\mathcal{R}_2^- =& -\frac{\tau}{u}\, e^{i\pi \frac{p}{2u\tau}[(\nu+s-1)^2-(\tau-\mu+2\theta+s+1)^2]} \\
& \sum_{s'=1}^{u-1} \sum_{r'=1}^{p-1} \sum_{\substack{n\in\mathbb{Z} \\ 1-2u\le s'+2n\le 0 \\ ur'-ps'>0}} e^{i\pi(\nu+s-1)(r'-\frac{p}{u}(s'+2n))+i\pi(\tau-\mu+2\theta+s+1)(r'-\frac{p}{u}s')} \\
&\qquad \times e^{-2i\pi n\tau(r'-\frac{p}{u}(s'+n))}\vartheta(2pu\tau, p\nu+ur'\tau-p(s'+2n)\tau) \ .
\end{aligned} \tag{5.2.27}$$

The terms $\mathcal{R}_1^+$ and $\mathcal{R}_2^+$ are obtained by using the set of changes in (5.2.24) except for a difference that is $\eta \to +\frac{\nu+s-1}{2}$. However before subtracting the $\mathcal{R}^+$ terms from their negative counterparts[3], we could simplify the Φ function in (5.2.26). To do so, we change,

$$\begin{aligned}
\tau &\to 2pu\tau \ , \\
\mu &\to 1-p\mu-u(r'-1)\tau-p(s'-2)\tau \ ,
\end{aligned}$$

in (3.4.7) and take,

$$m = p(2\theta+s+1)-1 \ .$$

This allows us to rewrite the mentioned Φ function in terms of $\Phi(2pu\tau, 1-p\mu-u(r'-1)\tau-p(s'-2)\tau)$, which has a term 1 in its second argument. To remove this

[3]We remind that in the last chapter the simplified Φ was implemented after the subtraction of *plus* terms form their *negative* counterparts, while here it is inserted before that, which is basically the same.

inelegant term, we use (3.4.10) and finally write the Φ function inside of formula (5.2.26) as,

$$\begin{aligned}
&\Phi(2pu\tau, p(2\theta+s+1) - p\mu - u(r'-1)\tau - p(s'-2)\tau) = \\
&\quad - \Big[e^{-\frac{i\pi}{2pu\tau}[p(2\theta+s+1)-1]^2 - \frac{i\pi}{pu\tau}[p(2\theta+s+1)-\frac{1}{2}] + \frac{i\pi}{u\tau}(2\theta+s+1)(p\mu+u(r'-1)\tau+p(s'-2)\tau)} \times \\
&\qquad \times \Phi(2pu\tau, p\mu + u(r'-2p-1)\tau + p(s'-2)\tau) \Big] \\
&\quad + \frac{i}{\sqrt{-2ipu\tau}} \sum_{j=1}^{p(2\theta+s+1)-1} e^{\frac{i\pi}{2pu\tau}[j^2 - 2pj(2\theta+s+1)] + \frac{i\pi j}{pu\tau}(p\mu+u(r'-1)\tau+p(s'-2)\tau)} . \qquad (5.2.28)
\end{aligned}$$

Inserting the right hand side of the formula above instead of Φ term in (5.2.26) produces two separate terms namely, $\mathcal{R}_{11}^-$ and $\mathcal{R}_{12}^-$. Including the $\mathcal{R}_2^-$ term, we therefore have three. Now if in a very similar way to what was done in the last chapter, we produce their *positive* counterparts and subtract them from the *negative* ones, a very long and tedious but precise calculation shows that [29],

$$\mathcal{R} = \Delta\mathcal{R}_{11} + \Delta\mathcal{R}_{12} + \Delta\mathcal{R}_2 \ , \qquad (5.2.29)$$

where,

$$\begin{aligned}
&\Delta\mathcal{R}_{11} = \mathcal{R}_{11}^- - \mathcal{R}_{11}^+ = \mathcal{N}_{s,\theta}(\tau,\nu,\mu) e^{i\pi p\frac{\mu^2}{2u\tau} - i\pi\frac{p}{u}\frac{\tau}{2}} \times \\
&\times \sum_{r'=1}^{2p-1} \sum_{s'=1}^{u} \sum_{\substack{n \\ 1\le s'+2n\le 2u}} e^{i\pi\frac{p}{u}(2\theta+s+1)(s'-2) - i\pi\frac{p}{u}(s'+2n-1)(s-1) + i\pi\frac{u\tau}{2p}(r'-\frac{p}{u}(s'+2n-1))^2} \\
&\qquad \times e^{i\pi(r'-\frac{p}{u}(s'+2n-1))\nu} \Phi(2pu\tau, p\mu - ur'\tau + p(s'-2)\tau) \Lambda_{r',s'+2n,u,p}(\tau,\nu). \qquad (5.2.30)
\end{aligned}$$

Remarkably this contains the same factor, namely $\mathcal{N}_{s,\theta}(\tau,\nu,\mu)$ introduced in (5.2.22) for leading (and its corrective) term,

$$\begin{aligned}
&\Delta\mathcal{R}_{12} = \mathcal{R}_{12}^- - \mathcal{R}_{12}^+ = \mathcal{N}_{s,\theta}(\tau,\nu,\mu) \frac{i e^{-i\pi\frac{p}{u}\frac{\tau}{2}}}{\sqrt{-2ipu\tau}} \sum_{j=0}^{p(2\theta+s+1)-2} e^{i\pi\frac{(p\mu+j)^2}{2pu\tau}} \\
&\qquad \times \sum_{r'=1}^{p-1} \sum_{s'=1}^{2u} e^{-i\pi\frac{p}{u}(2\theta+s+1)([s']_2+2) - i\pi\frac{p}{u}(s'-1)(s-1) + i\pi\frac{u\tau}{2p}(r'-\frac{p}{u}(s'-1))^2} \\
&\quad \times e^{-2i\pi j\frac{ur'+p([s']_2+2)}{2pu}} e^{i\pi(r'-\frac{p}{u}(s'-1))\nu} \Lambda_{r',s',u,p}(\tau,\nu) \sum_{n=1}^{u} e^{2i\pi\frac{n}{u}(j+p(2\theta+s+1))}, \qquad (5.2.31)
\end{aligned}$$

and eventually,

$$\Delta\mathcal{R}_2 = \mathcal{R}_2^- - \mathcal{R}_2^+ = \mathcal{N}_{s,\theta}(\tau,\nu,\mu) \sum_{r'=1}^{p-1} \sum_{s'=1}^{2u} \sum_{\substack{b\in\mathbb{Z} \\ 2b+s'-1\geq 1 \\ p(2b+s'-1)<ur'}} e^{-i\pi\frac{p}{u}(s-1)(s'-1)-i\pi\frac{p}{u}(2\theta+s+1)(2b+s')-2i\pi\frac{p}{u}b^2\tau}$$
$$\times\, e^{i\pi(2b+1)(r'-\frac{p}{u}s')\tau-i\pi(r'-\frac{p}{u}(2b+s'))\mu+i\pi(r'-\frac{p}{u}(s'-1))\nu}\Lambda_{r',s',u,p}(\tau,\nu) \quad (5.2.32)$$

Here, the very important term $\Delta\mathcal{R}_{11}$ in (5.2.30), in which the only contribution of the Φ function for the S modular transformation of $\widehat{s\ell}(2|1)$ characters is observed, is in its final proper form and does not need to be simplified any more. Nevertheless we still need to deal with $\Delta\mathcal{R}_{12}$ and $\Delta\mathcal{R}_2$ as follows.

We clearly see that the last sum in (5.2.31), sets j equal to $-p(2\theta+s+1)+ua$, $a\in\mathbb{Z}$, which has a remarkable effect that all occurrences of $[s']_2$, can be replaced with s', and after some simple rearrangements, we find,

$$\Delta\mathcal{R}_{12} = -\sqrt{\tfrac{-i\tau}{2pu}} \sum_{a} \sum_{r'=1}^{p-1} \sum_{s'=1}^{2u} e^{i\pi a(1+\frac{\nu}{\tau}+\frac{\mu}{\tau}-(\theta+1)\frac{2}{\tau})}$$
$$\times\, e^{i\pi\frac{pu}{2\tau}(\frac{\nu+s-1}{u}-\frac{a}{p})^2+i\pi u(\frac{\nu+s-1}{u}-\frac{a}{p})(r'-\frac{p}{u}(s'-1))+i\pi\frac{u\tau}{2p}(r'-\frac{p}{u}(s'-1))^2}\Lambda_{r',s',u,p}(\tau,\nu), \quad (5.2.33)$$

which significantly its summand is minus the one in (5.2.15) we obtained in Step 1. To us, this fact alone, justifies all the efforts to manipulate this involved transformation so that Λ functions arise AND it shows this general appearance. However as we have not yet succeeded to represent these two terms properly unified (or more simplified), we just continue as,

$$\mathcal{N}_{s,\theta}(\tau,\nu,\mu)\Omega = \Gamma + \Delta\mathcal{R}_{12} = \sqrt{\tfrac{-i\tau}{2pu}}\,\Big(\sum_{a=1}^{r-1}\sum_{r'=1}^{2p}\sum_{s'=1}^{u} - \sum_{a=l_1}^{l_2}\sum_{r'=1}^{p-1}\sum_{s'=1}^{2u}\Big)\, e^{i\pi a(1+\frac{\nu}{\tau}+\frac{\mu}{\tau}-(\theta+1)\frac{2}{\tau})}$$
$$\times\, e^{i\pi\frac{pu}{2\tau}(\frac{\nu+s-1}{u}-\frac{a}{p})^2+i\pi u(\frac{\nu+s-1}{u}-\frac{a}{p})(r'-\frac{p}{u}(s'-1))+i\pi\frac{u\tau}{2p}(r'-\frac{p}{u}(s'-1))^2}\Lambda_{r',s',u,p}(\tau,\nu), \quad (5.2.34)$$

where $l_1 = int(\frac{p}{u}(2\theta+s+1))$ and $l_2 = int(\frac{2p}{u}(2\theta+s+1)-\frac{2}{u})$.

We are now approaching to finish the Step 2 of our calculations by reminding the way we combined two terms $\mathcal{G}_2$ in (4.2.66) and $\Delta\mathcal{H}_2$ in (4.2.64) for $N=2$ non-unitary characters. In fact $\mathcal{G}_2$ is analogous to $\mathcal{N}_{s,\theta}(\tau,\nu,\mu)\mathcal{Q}'$ and $\Delta\mathcal{H}_2$ one for $\Delta\mathcal{R}_2$. Thus to avoid repeating the same combination procedure, we only write the final precise result as,

$$\mathcal{N}_{s,\theta}(\tau,\nu,\mu)\tilde{\mathcal{Q}} = \mathcal{N}_{s,\theta}(\tau,\nu,\mu)\mathcal{Q}' + \Delta\mathcal{R}_2 =$$

$$= -\mathcal{N}_{s,\theta}(\tau,\nu,\mu)\sum_{r'=1}^{p-1}\sum_{b=1}^{u}\sum_{s'=2b-1}^{2u} e^{-i\pi\frac{p}{u}(s-1)(s'-1)+i\pi\frac{p}{u}(2\theta+s+1)(s'-2b)-2i\pi\frac{p}{u}b^2\tau}$$

$$\times\, e^{i\pi(2b-1)(r'+\frac{p}{u}s')\tau-i\pi(r'+\frac{p}{u}(s'-2b))\mu+i\pi(r'-\frac{p}{u}(s'-1))\nu-2i\pi(s'-1)r'\tau}\Lambda_{r',s',u,p}(\tau,\nu). \quad (5.2.35)$$

Therefore in accordance with (5.2.3) we hereby present our derivation of the S-transform of $\psi_{r,s,u,p;\theta}(\tau,\nu,\mu)$ by,

$$\boxed{\begin{aligned}
&\psi_{r,s,u,p;\theta}(\tfrac{-1}{\tau},\tfrac{\nu}{\tau},\tfrac{\mu}{\tau}) = \mathcal{N}_{s,\theta}(\tau,\nu,\mu)e^{-\frac{i\pi}{\tau}(r-1)(\nu+\mu-2(\theta+1))+i\pi(r-1)}\\
&\times\Bigg[\sum_{b=0}^{u-1}\sum_{s'=1}^{u} e^{-i\pi\frac{p}{u}(s-1)(s'-1)+i\pi\frac{p}{u}(2\theta+s+1)(2b+s')-2i\pi\frac{p}{u}b^2\tau}\\
&\quad\times e^{-i\pi\frac{p}{u}(2b+1)s'\tau-i\pi\frac{p}{u}(2b+s')\mu-i\pi\frac{p}{u}(s'-1)\nu}\psi_{1,s',u,p;b}(\tau,\nu,\mu-1-\tau)\\
&\quad+e^{i\pi p\frac{\mu^2}{2u\tau}-i\pi\frac{p}{u}\frac{\tau}{2}}\sum_{r'=1}^{2p-1}\sum_{s'=1}^{u}\sum_{\substack{n\\1\le s'+2n\le 2u}} e^{i\pi\frac{p}{u}(2\theta+s+1)(s'-2)-i\pi\frac{p}{u}(s'+2n-1)(s-1)}\\
&\quad\times e^{i\pi\frac{u\tau}{2p}(r'-\frac{p}{u}(s'+2n-1))^2+i\pi(r'-\frac{p}{u}(s'+2n-1))\nu}\Phi(2pu\tau,p\mu-ur'\tau+p(s'-2)\tau)\\
&\quad\times\Lambda_{r',s'+2n,u,p}(\tau,\nu)+\tilde{\mathcal{Q}}+\Omega\Bigg]. \quad (5.2.36)
\end{aligned}}$$

Where $\tilde{\mathcal{Q}}$ and Ω are respectively given by (5.2.35) and (5.2.34).
Furthermore we could insert $\psi_{r,s,u,p;\theta}(\frac{-1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})$ from transformation law above, to (5.1.14) with considering (5.1.17), to acquire the S-transform of $\chi_{r,s,u,p;\theta}(q,x,y)$ the non-unitary $\widehat{s\ell}(2|1)$ characters, as we did for $N=2$ characters in (4.2.71)-(4.2.75).

Step 3: S-transform of unitary case

In the last chapter without using the facilities of Appell functions, we had a different way to achieve the S-transform of unitary $N=2$ characters. This way can be entirely repeated to compute the S-transform of unitary $\widehat{s\ell}(2|1)$ ones, to be compared with (5.2.36) for $r=p=1$ i.e. the unitary $\widehat{s\ell}(2|1)$ case. Although we do not intend to do it again, we still wish to show the unitary case by using (5.2.36).

Putting $r = p = 1$ in (5.2.36) and using (C.2.3) easily implies the cancellation of any term which contains the Λ function [4], and one is only left with,

$$\psi_{s,u;\theta}(\tfrac{-1}{\tau}, \tfrac{\nu}{\tau}, \tfrac{\mu}{\tau}) = \mathcal{N}_{s,\theta}(\tau,\nu,\mu) \sum_{b=0}^{u-1} \sum_{s'=1}^{u} e^{-\frac{i\pi}{u}(s-1)(s'-1)+\frac{i\pi}{u}(2\theta+s+1)(2b+s')-\frac{2i\pi}{u}b^2\tau}$$
$$\times\, e^{-\frac{i\pi}{u}(2b+1)s'\tau-\frac{i\pi}{u}(2b+s')\mu-\frac{i\pi}{u}(s'-1)\nu} \psi_{s',u;b}(\tau,\nu,\mu-1-\tau), \quad (5.2.37)$$

where we have defined,

$$\psi_{1,s,u,1;\theta}(\tau,\nu,\mu) \equiv \psi_{s,u;\theta}(\tau,\nu,\mu). \quad (5.2.38)$$

It only remains to say that, $\psi_{r,s,u,p;\theta}(\tau,\nu,\mu)$ functions are also T-transformed as easy as $\varphi_{r,s,u,p;\theta}(\tau,\nu)$ functions we saw in (4.2.70). So,

$$\psi_{r,s,u,p;\theta}(\tau+1,\nu,\mu) = \psi_{r,s,u,p;\theta}(\tau,\nu,\mu)\ , \quad (5.2.39)$$

implies a very trivial T-transformation for $\widehat{s\ell}(2|1)$ characters.

[4] This easy obtaining of unitary transformation is not just an accident. In fact this was the main reason to try to establish (5.2.3).

Chapter 6

$N = 2$ characters as the residue of $\widehat{s\ell}(2|1)$ characters

6.1 Introduction

During the last two chapters, we tried to obtain the modular transformations of non-unitary $N = 2$ and $\widehat{s\ell}(2|1)$ characters via higher-level Appell functions. We dealt also with consistency checks, while reducing the non-unitary cases to unitary ones which could be achieved without using the Appell functions. Therefore to us, it confirmed the validity of using the valuable properties of these functions, however in this chapter we wish to dedicate another way as, not only a reconfirmation but also an interesting point alone, which links $\widehat{s\ell}(2|1)$ and $N = 2$ characters by residue calculation.

We find in [6], [24] and [45] description of some successful efforts to connect respectively, $\widehat{s\ell}(2)$ characters to Virasoro ones, $\widehat{osp}(2,2)$ to $N = 2$ unitary, and $\widehat{osp}(1,2)$ to $N = 1$ unitary, by using the residue calculation. Hereby, we would also like to present our residue calculation of non-unitary $\widehat{s\ell}(2|1)$ characters by which, we obtain an expression including non-unitary $N = 2$ characters.

6.2 The residue of non-unitary $\widehat{s\ell}(2|1)$ characters

We firstly start by recalling (5.1.14) along with (5.1.17) to represent,

$$\chi_{(r,s,u,p;\theta)}(q,x,y) = \left\{ y^{-\frac{p}{u}\theta} q^{-\frac{p}{u}\theta^2} q^{[(r-1)-(s+1)\frac{p}{u}]\theta} \right\} \times$$
$$\times \left\{ \frac{\vartheta_{(1,0)}(q,x^{\frac{1}{2}}y^{\frac{1}{2}})\vartheta_{(1,0)}(q,x^{\frac{1}{2}}y^{-\frac{1}{2}})}{\vartheta_{(1,1)}(q,x)\prod_{i\geq 1}(1-q^i)^3} x^{\frac{r-1}{2}-\frac{s-1}{2}\frac{p}{u}} y^{\frac{r-1}{2}-\frac{s+1}{2}\frac{p}{u}} q^{r-1-\frac{sp}{u}} \right\} \psi_{(r,s,u,p)}(q,x,yq^{2\theta}), \tag{6.2.1}$$

where $k = \frac{p}{u} - 1$ and the range of labels are given by (5.1.16).

The two braces contain all the possible prefactors, however it is just the first one which includes all θ-dependent factors. Now, we write $\psi_{(r,s,u,p)}(q,x,yq^{2\theta})$ in (6.2.1) as,

$$\psi_{(r,s,u,p)}(q,x,yq^{2\theta}) = \tilde{\psi}_{(u,p)}(q,\tilde{x},\tilde{y}q^{2\theta}), \tag{6.2.2}$$

where,

$$\tilde{x} = xq^{1-s} \ , \ \tilde{y} = yq^{s+1-2\alpha u} \ , \ \alpha = \frac{r-1}{2p} \ . \tag{6.2.3}$$

This leads the following simplification,

$$\tilde{\psi}_{(u,p)}(q,\tilde{x},\tilde{y}q^{2\theta}) = q^{-up\alpha^2}\tilde{x}^{-p\alpha} \sum_{m\in\mathbb{Z}-\alpha} q^{upm^2} [\frac{\tilde{x}^{-pm}}{1+\tilde{x}^{-\frac{1}{2}}\tilde{y}^{-\frac{1}{2}}q^{mu-\theta}} - \frac{\tilde{x}^{pm}}{1+\tilde{x}^{\frac{1}{2}}\tilde{y}^{-\frac{1}{2}}q^{mu-\theta}}]. \tag{6.2.4}$$

Moreover as we see in (6.2.1) the $\widehat{s\ell}(2|1)$ character formula has simple poles in the variable x due to the presence of $\vartheta_{(1,1)}(q,x)$ in the denominator[1]. They occur when $x = q^n \ , n \in \mathbb{Z}/\{u\mathbb{Z}+s-1\}$. Indeed when $x = q^{au+s-1}$ for any integer a, the numerator of $\tilde{\psi}_{(u,p)}(q,\tilde{x},\tilde{y}q^{2\theta})$ vanishes and therefore removes the pole.

Proof : $x = q^{au+s-1} \Rightarrow \tilde{x} = q^{au}$ so the expression (6.2.4) goes to ,

$$\begin{aligned}\tilde{\psi}_{(u,p)}(q,\tilde{x},\tilde{y}q^{2\theta}) &= q^{-up\alpha^2}\tilde{x}^{-p\alpha} \sum_{m\in\mathbb{Z}-\alpha} [\frac{q^{up(m^2-am)}}{1+q^{u(m-a/2)-\theta}\tilde{y}^{-\frac{1}{2}}} - \frac{q^{up(m^2+am)}}{1+q^{u(m+a/2)-\theta}\tilde{y}^{-\frac{1}{2}}} \ ] \\ &= q^{-up(\alpha^2+a^2/4)}\tilde{x}^{-p\alpha} \sum_{m\in\mathbb{Z}-\alpha} [\frac{q^{up(m-a/2)^2}}{1+q^{u(m-a/2)-\theta}\tilde{y}^{-\frac{1}{2}}} - \frac{q^{up(m+a/2)^2}}{1+q^{u(m+a/2)-\theta}\tilde{y}^{-\frac{1}{2}}} \ ] \ .\end{aligned}$$

[1]See the production formula of $\vartheta_{(1,1)}(q,x)$ in (2.2.9).

Now by shifting $m \to m+a$ in the first fraction, both fractions become equal and the series vanishes. So all poles are of the form : $x = q^{au+s-1-\gamma}$, or $\tilde{x} = q^{au-\gamma}$ where $1 \leq \gamma \leq u-1$. The corresponding residues are given by,

$$\tilde{\psi}_{(u,p)}(q,\tilde{x},\tilde{y}q^{2\theta}) = q^{-up(\alpha^2+a^2/4)}\tilde{x}^{-p\alpha}\times$$
$$\times \sum_{m\in\mathbb{Z}-\alpha}[\frac{q^{up(m-a/2)^2+mp\gamma}}{1+q^{u(m-a/2)+\gamma/2-\theta}\tilde{y}^{-\frac{1}{2}}} - \frac{q^{up(m+a/2)^2-mp\gamma}}{1+q^{u(m+a/2)-\gamma/2-\theta}\tilde{y}^{-\frac{1}{2}}}]. \quad (6.2.5)$$

We now show how these residues are related to $N=2$ superconformal characters. Shifting $m \to m-a$ in the second fraction in (6.2.5) we get,

$$\tilde{\psi}_{(u,p)}(q,\tilde{x},\tilde{y}q^{2\theta}) = q^{-up\alpha^2}\tilde{x}^{-p\alpha}q^{-upa^2/4+p\gamma a/2}\times$$
$$\times \sum_{m\in\mathbb{Z}-\alpha}[\frac{q^{up(m-a/2)^2+(m-a/2)p\gamma}}{1+q^{u(m-a/2)+\gamma/2-\theta}\tilde{y}^{-\frac{1}{2}}} - \frac{q^{up(m-a/2)^2-(m-a/2)p\gamma}}{1+q^{u(m-a/2)-\gamma/2-\theta}\tilde{y}^{-\frac{1}{2}}}]. \quad (6.2.6)$$

Introducing $\tilde{x}_\gamma = q^{-\gamma}$ and $\beta = \alpha + \text{a}/2$, gives $q^{-up\alpha^2}\tilde{x}^{-p\alpha}q^{-upa^2/4+p\gamma a/2} = q^{-up\beta^2}\tilde{x}_\gamma^{-p\beta}$. Then (6.2.6) becomes,

$$\tilde{\psi}_{(u,p)}(q,\tilde{x},\tilde{y}q^{2\theta}) = q^{-up\beta^2}\tilde{x}_\gamma^{-p\beta}\sum_{l\in\mathbb{Z}-\beta} q^{upl^2}[\frac{\tilde{x}_\gamma^{-lp}}{1+q^{ul-\theta}\tilde{x}_\gamma^{-\frac{1}{2}}\tilde{y}^{-\frac{1}{2}}} - \frac{\tilde{x}_\gamma^{lp}}{1+q^{ul-\theta}\tilde{x}_\gamma^{\frac{1}{2}}\tilde{y}^{-\frac{1}{2}}}]. \quad (6.2.7)$$

Remembering the series part of $N=2$ superconformal characters, we write[2],

$$\varphi_{(r_N,s_N,u,p;\theta_N)}(q,z) =$$
$$\sum_{m\in\mathbb{Z}} q^{m^2up-mu(s_N-1)}[\frac{q^{mpr_N}}{1+z^{-1}q^{mu+\theta_N}} - q^{r_N(s_N-1)}\frac{q^{-mpr_N}}{1+z^{-1}q^{mu+\theta_N-r_N}}] . \quad (6.2.8)$$

We similarly write $\varphi_{(r_N,s_N,u,p;\theta_N)}(q,z)$ in equation above as,

$$\varphi_{(r_N,s_N,u,p;\theta_N)}(q,z) = \tilde{\varphi}_{(u,p;\theta_N)}(q,x_N,y_N) , \quad (6.2.9)$$

where $x_N = q^{-r_N}$, $y_N = z^2q^{r_N-2\alpha_N u}$ and $\alpha_N = \frac{s_N-1}{2p}$, which leads

$$\tilde{\varphi}_{(u,p;\theta_N)}(q,x_N,y_N) =$$
$$q^{-up\alpha_N^2}x_N^{-p\alpha_N}\sum_{m\in\mathbb{Z}-\alpha_N} q^{upm^2}[\frac{x_N^{-mp}}{1+q^{um}x_N^{-\frac{1}{2}}y_N^{-\frac{1}{2}}q^{\theta_N}} - \frac{x_N^{mp}}{1+q^{um}x_N^{\frac{1}{2}}y_N^{-\frac{1}{2}}q^{\theta_N}}]. \quad (6.2.10)$$

[2]To avoid confusion we have put subscripts of N for labels of $N=2$ characters which are being used for $\widehat{s\ell}(2|1)$ theory as well.

Comparing the two formulas (6.2.7) and (6.2.10) we see that,

$\tilde{x}_\gamma = x_N \;\Rightarrow\; q^{-\gamma} = q^{-r_N} \;\Rightarrow\; r_N = \gamma = au + s - 1 - n,$

$\tilde{y}^{-\frac{1}{2}} = y_N^{-\frac{1}{2}} \;\Rightarrow\; y = z^2 q^{-(n+2)},$

$\beta = \alpha_N \Rightarrow \alpha + a/2 = \alpha_N \;\Rightarrow\; s_N = r + ap,$

and $\theta = -\theta_N$.

Now remembering (6.2.2) and using all the substitutions above to convert the rest of terms in formula (6.2.1) i.e. prefactors term by term. Hence one obtains,

$$\begin{aligned}
y^{-\frac{p}{u}\theta} &\;\to\; y^{\frac{p}{u}\theta_N} \;\to\; z^{2\frac{p}{u}\theta_N} q^{-(n+2)\frac{p}{u}\theta_N},\\
q^{-\frac{p}{u}\theta^2} &\;\to\; q^{-\frac{p}{u}\theta_N^2},\\
q^{[(r-1)-(s+1)\frac{p}{u}]\theta} &\;\to\; q^{-[s_N-1-(n+2+r_N)\frac{p}{u}]\theta_N},\\
\vartheta_{(1,0)}(q, x^{\frac{1}{2}} y^{\frac{1}{2}}) &\;\to\; z\,\vartheta_{(1,0)}(q,z),\\
\vartheta_{(1,0)}(q, x^{\frac{1}{2}} y^{-\frac{1}{2}}) &\;\to\; q^{-n^2/2-3n/2-1}\, z^{n+2}\, \vartheta_{(1,0)}(q,z),
\end{aligned}$$

as well as using the identity,

$$\vartheta'_{(1,1)}(q,x)|_{x=q^n} = (-1)^n q^{-n^2/2-3n/2} \prod_{i\geq 1}(1-q^i)^3,$$

and substituting the new amounts of "y" and "x" and finally putting everything together we obtain,

$$\begin{aligned}
res\;\; \chi_{(r,s,u,p;\theta)}(q,x,y)|_{x=q^n} &= q^{-\frac{p}{u}(\theta_N^2-\theta_N)+\theta_N\frac{p}{u}(r_N-1)-\theta_N(s_N-1)}\, z^{2\theta_N\frac{p}{u}+(s_N-1)-\frac{p}{u}(r_N-1)} \times \\
&\frac{\vartheta_{(1,0)}\,(q,z)}{\prod_{i\geq 1}(1-q^i)^3} \times \varphi_{(r_N,s_N,u,p;\theta_N)}(q,z) \frac{(-1)^n \vartheta_{(1,0)}\,(q,z_N)}{\prod_{i\geq 1}(1-q^i)^3} z^{-(n+3)(\frac{p}{u}-1)}\, q^{\frac{p}{u}(n+1)-1} \;.
\end{aligned} \tag{6.2.11}$$

The first four factors above construct the $\omega_{(r_N,s_N,u,p;\theta_N)}(q,z)$ introduced in (4.1.7), which is an $N=2$ character at central charge $c = 3(1-2p/u)$ labelled by r_N and s_N. However both labels are not in their fundamental ranges yet. Thus we write those in terms of r, s and θ and continue the equality (6.2.11) as,

$$\begin{aligned}
&res\;\; \chi_{(r,s,u,p;\theta)}(q,x,y)|_{x=q^n} = \\
&\qquad \omega_{(au+s-1-n,r+ap,u,p;-\theta)}(q,z) \frac{(-1)^n \vartheta_{(1,0)}\,(q,z)}{\prod_{i\geq 1}(1-q^i)^3}\, z^{-(n+3)(\frac{p}{u}-1)}\, q^{\frac{p}{u}(n+1)-1},
\end{aligned} \tag{6.2.12}$$

Using $\omega_{(ku+r,s,u,p;\theta)} = \omega_{(r,s-kp,u,p;\theta)}$, $k \in \mathbb{Z}$ and putting $z = q^{(1+n/2)}y^{\frac{1}{2}}$ in formula above we finally arrive at,

$$\boxed{\begin{aligned} &res\ \chi_{(r,s,u,p;\theta)}(q,x,y)|_{x=q^n} = \\ &\omega_{(s-1-n,r,u,p;-\theta-1)}(q,q^{n/2}y^{\frac{1}{2}})\frac{(-1)^n\vartheta_{(1,0)}\,(q,q^{n/2}y^{\frac{1}{2}})}{\prod_{i\geq 1}(1-q^i)^3}\,y^{\frac{1}{2}(n+1)(1-\frac{p}{u})}q^{n\frac{p}{u}+\frac{n}{2}(n+3)(1-\frac{p}{u})}, \end{aligned}} \tag{6.2.13}$$

in which identities (4.1.4) and (2.2.14) to deal with the twist θ have been used. Studying two special cases of $n = 0, -1$ will be of a great benefit in the future. Therefore inserting $n = -1$ in description above gives,

$$res\ \ \chi_{(r,s,u,p;\theta)}(q,x,y)|_{x=q^{-1}} \;=\; -\omega_{(s,r,u,p;-\theta)}(q,(qy)^{\frac{1}{2}})\frac{\vartheta_{(1,0)}\,(q,(qy)^{\frac{1}{2}})}{\prod_{i\geq 1}(1-q^i)^3}\,y^{1-\frac{p}{u}}q^{-\frac{p}{u}}. \tag{6.2.14}$$

And for $n = 0$ the residue formula in (6.2.13) becomes,

$$res\ \chi_{(r,s,u,p;\theta)}(q,x,y)|_{x=q^0=1} = \omega_{(s-1,r,u,p;-\theta-1)}(q,y^{\frac{1}{2}})\frac{\vartheta_{(1,0)}\,(q,y^{\frac{1}{2}})}{\prod_{i\geq 1}(1-q^i)^3}y^{\frac{1}{2}(1-\frac{p}{u})}. \tag{6.2.15}$$

• Now to perform a consistency check for all the calculations done via higher-level Appell functions, we believe one could use the general description (6.2.13) to link the S-transform of admissible $\widehat{s\ell}(2|1)$ characters, with $N = 2$ ones, obtained in chapters 5 and 4. However it still well worth to use one of the special cases above, e.g. (6.2.15) to deal with that. In order to use (6.2.15) one should consider the following prescription, happens after operating the residue calculation for $\widehat{s\ell}(2|1)$ characters (for $x = q^0 = 1$),

$$\begin{aligned} \widehat{s\ell}(2|1) &\overset{residue}{\longrightarrow} \quad N=2 \\ \nu &\longrightarrow \quad 0 \\ r &\longrightarrow \quad s \\ s &\longrightarrow \quad r+1 \\ \mu &\longrightarrow \quad 2\nu \\ \theta &\longrightarrow \quad -\theta-1 \end{aligned} \tag{6.2.16}$$

Therefore the prescription above could be used to simply translate a $\widehat{s\ell}(2|1)$ expression to its $N = 2$ equivalent. For example using this translation for interpretation of $\psi_{r,s,u,p;\theta}(\tau,\nu,\mu)$ in terms of $\mathcal{K}_p$'s which reads,

$$
\begin{aligned}
&\psi_{r,s,u,p;\theta}(\tau,\nu,\mu) = \\
&\quad = \mathcal{K}_{2p}(u\tau, -\tfrac{\nu}{2} - \tfrac{u(r-1)}{2p}\tau + \tfrac{s-1}{2}\tau, \tfrac{1}{2} - \tfrac{\mu}{2} + [\tfrac{u(r-1)}{2p} - \tfrac{s+1}{2} + \theta]\tau) \\
&\qquad - e^{2i\pi(s-1)(r-1)\tau - 2i\pi(r-1)\nu} \\
&\qquad \times \mathcal{K}_{2p}(u\tau, \tfrac{\nu}{2} - \tfrac{u(r-1)}{2p}\tau - \tfrac{s-1}{2}\tau, \tfrac{1}{2} - \tfrac{\mu}{2} + [\tfrac{u(r-1)}{2p} - \tfrac{s+1}{2} + \theta]\tau)\ , \qquad (6.2.17)
\end{aligned}
$$

immediately gives (4.1.12), which is written in terms of Appell functions as well. Therefore it is not unreasonable to expect that one could use the prescription (6.2.16) to go from (5.2.36) to the S transformation of $\varphi_{r,s,u,p}(\tau,\nu)$ we obtained for $N = 2$ characters[3]. The prescription (6.2.16) can be used to obtain the S-transform $\omega_{r,s,u,p;\theta}(-\frac{1}{\tau},\frac{\nu}{\tau})$ from the S-transform $\chi_{r,s,u,p;\theta}(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau})$ and one should not also forget to impose the extra factor $\vartheta_{(1,0)}(q,y^{\frac{1}{2}})y^{\frac{1}{2}(1-\frac{p}{u})}/\prod_{i\geq 1}(1-q^i)^3$ appearing in (6.2.15). The derivation has been carried out. It is tedious but conceptually straightforward, and we do not reproduce the details here.

[3]We must put $s = 1$ for φ (for leading terms and corrective terms separately), since our S-transform derivations for ψ have been done for $r = 1$.

Chapter 7

Conclusions

Higher-level Appell functions have proven to be an extremely useful mathematical tool in our work. They are born from a generalisation of the complex function of three variables introduced by M.P. Appell about 150 years ago in the context of the study of elliptic functions of the third kind. This generalisation has been suggested by the problem we wanted to solve initially, namely the derivation of the modular transformations of characters of infinite-dimensional algebras, and in particular the $N = 2$ superconformal algebra and the affine superalgebra $\widehat{s\ell}(2|1)$, when the associated representations are *admissible.* In the two cases we treated, the mathematical manifestation of 'strict' admissibility is the fact that the parameter p is different from 1 in the central charge $c = 3(1 - \frac{2p}{u})$ ($N = 2$) and the level $k = \frac{p}{u} - 1$ ($\widehat{s\ell}(2|1)$), with p and u coprime. By 'strict' we mean that we exclude the $N = 2$ representations which are minimal and unitary on the one hand, and the $\widehat{s\ell}(2|1)$ representations with level $k = \frac{1}{u} - 1$ on the other. Although non-integrable, the characters of the latter have S modular transformations which yield linear combinations involving the initial finite number of characters [24,25,43]. The complication of strict admissibility is that the characters are not quasiperiodic in one of the parameters called 'spectral flow'. When shifted by u in a given character, this parameter θ does lead to a function which can be split into the initial character modulo a 'corrective' term involving Λ-functions (4.1.11). The corrective terms in non-quasiperiodic behaviours are a signal that the notion of invariance under S can only make sense if one considers the character under investigation together with a number (depending on p) of Λ functions (C.2.1), which seem to be associated with the characters of a related algebraic structure, although we have no proof of this at the moment. A remarkable feature, and certainly not a coincidence, is that these Λ functions appear both in non-quasiperiodicity formulas and S modular transformation statements, as the "violating" terms of quasiperiodic formulas or unitary S-transformations.

In dealing with S-transform of characters, the non-trivial part is (4.1.6) and (5.1.15). Semikhatov and Tipunin manufactured the higher-level Appell functions (1.19) which allow to rewrite these non-trivial factors of characters as differences of Appell functions (4.1.12) and (5.2.16). So in the strictly admissible case, Appell functions play the role the ϑ functions play when the characters are periodic in the spectral flow.

Our thesis makes clear the structure behind the S-transform of the higher-level Appell functions (1.21). As already explained in the introduction, the key is not to consider the level p Appell function on its own, but to build a $(p+1)$-dimensional vector $\mathbb{K}_p(\tau,\nu,\mu)$ where the extra p components are the functions $\vartheta(p\tau,p\nu+n\tau)$, $n=0,..,p-1$. Although the Appell function $\mathcal{K}_p(\tau,\nu,\mu)$ is not mapped into itself under S, the vector $\mathbb{K}_p(\tau,\nu,\mu)$ enjoys the invariance property (1.35). One expects a similar structure for the characters. However, these being written as *differences* of Appell functions, the corrective terms should be *differences* of ϑ functions. The difficulty is to identify whether these differences are always of the same structure. This is the reason why we introduced the Λ functions, and have made all efforts to re-express the corrective terms exclusively in terms of those and the Φ function already present in the S-transform of $\mathcal{K}_p$. We have succeeded in that goal, but several highly non-trivial identities and manipulations are involved, and we suppose that the most satisfactory structure of the corrective terms has not been reached yet (e.g. formula (5.2.34)). However we are trying to present the best of appearances of all the corrective terms in the future publication. All relevant identities involving higher-level Appell functions have been classified in Chapter 2, while the implementation of those to obtain the crucial relations of modular transformations of higher-level Appell functions has been done in Chapter 3. The very important function called

$\Phi(\tau,\mu)$ arises while dealing with the S transformation of Appell functions, and we have had a thorough study of it at the end of that chapter. In Chapters 4 and 5, we came back to the original problem i.e. calculating the S modular transformation of non-unitary $N = 2$ and $\widehat{s\ell}(2|1)$ characters. Our first goal was to reproduce an essential 'leading term' as outcome of these calculations —which we achieved—with as few corrective terms as possible, all in terms of Λ functions. We were helped in that by several mathematical tricks we collect in appendices.

We were able to carry on one consistency check on our formula (4.2.75), namely we re-calculated with separate techniques (without using the higher-level Appell functions) the well-known S-transform of unitary minimal $N = 2$ characters, and compared successfully the result obtained with the formula (4.2.75) when $p = 1$. We also paved the way to another consistency check by obtaining the admissible $N = 2$ characters as residues of admissible $\widehat{s\ell}(2|1)$ characters. Consistency requires that the general S-transform formula for $\widehat{s\ell}(2|1)$ characters should yield, when residues are taken, the formula (4.2.75).

The current work is in preparation to be published. The techniques developed and tested in our thesis should be applicable in a wide range of admissible cases. An outstanding one is the behaviour of $N = 4$ superconformal characters first constructed by [44].

Appendix A

A.1 Periodicity properties of $\mathcal{K}_p$

• To show the quasi-periodicity of higher-level Appell functions in their second argument, (2.4.7), we simply start to write,

$$\mathcal{K}_{(p)}(q, xq^n, y) = \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m^2p}{2}} x^{mp} q^{mnp}}{1 - xyq^{m+n}}, \tag{A.1.1}$$

and change $m \to m - n$ to obtain

$$\begin{aligned}
\mathcal{K}_{(p)}(q, xq^n, y) &= \sum_{m\in\mathbb{Z}} \frac{q^{\frac{(m-n)^2p}{2}} x^{(m-n)p} q^{(m-n)np}}{1 - xyq^m} , \\
&= \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m^2p}{2} - mnp + n^2p2} x^{(m-n)p} q^{mnp} q^{-n^2p}}{1 - xyq^m} , \\
&= q^{-n^2p} x^{-np} \mathcal{K}_{(p)}(q, x, y) ,
\end{aligned}$$

which is the right-hand side of (2.4.7).

• Now we try to prove (2.4.9) for the case $n \in -\mathbb{N}$,(the case $n \in \mathbb{N}$ can easily be shown using the same technique). We start from right-hand side of (2.4.9) and write,

$$q^{\frac{n^2p}{2}} y^{np} \mathcal{K}_{(p)}(q, x, y) = \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m^2p}{2} + \frac{(m+n)^2p}{2}} y^{np} x^{(m+n)p}}{1 - xyq^{(m+n)}} . \tag{A.1.2}$$

Meanwhile,

$$\begin{aligned}\sum_{j=pn}^{-1} x^j\, y^j q^{nj}\theta(q^p, x^p q^j) &= \sum_{m\in\mathbb{Z}}\sum_{j=pn}^{-1}(xyq^n)^j q^{\frac{m^2p}{2}} x^{mp} q^{mj}\\ &= \sum_{m\in\mathbb{Z}} q^{\frac{m^2p}{2}} x^{mp}\sum_{j=pn}^{-1}(xyq^{m+n})^j\\ &= \sum_{m\in\mathbb{Z}} q^{\frac{m^2p}{2}} x^{mp}\frac{x^{np}y^{np}q^{(m+n)np}-1}{1-xyq^{m+n}}\ . \qquad \text{(A.1.3)}\end{aligned}$$

Where in the last line, lemma (2.4.14) has been used. Now by subtracting (A.1.3) from (A.1.2) one arrives at

$$\sum_{m\in\mathbb{Z}}\frac{q^{\frac{m^2p}{2}}x^{mp}}{1-xyq^{m+n}} = \mathcal{K}_{(p)}(q,x,yq^n)\ , \qquad \text{(A.1.4)}$$

which is the left-hand side of (2.4.9).

All other periodicity properties in subsection 2.4.2 can be derived in a similar way.

A.2 Period increasing statement for higher-level Appell functions

For Appell functions at arbitrary level p (not necessarily even), but when $(u,p)=1$ still, we have the following relation

$$\begin{aligned}\mathcal{K}_{(p)}(q,x,y) = \sum_{s=0}^{u-1}\sum_{b=0}^{u-1} x^{ps}\, y^{pb}\, q^{\frac{s^2-b^2}{2}p}\,\mathcal{K}_{(p)}(q^{u^2}, x^u\, q^{su}, y^u\, q^{-bu})&\\ +\sum_{\substack{r=1\\ ur-ps\geq 1}}^{p-1}\sum_{s=1}^{u-1} x^{ur}\, y^{ur-ps}\, q^{urs-\frac{ps^2}{2}}\theta(q^p, x^p\, q^{ur}).&\end{aligned}$$

Its proof is straightforward but requires, as for the proof of (2.4.20), the derivation of the crucial identity (2.4.17), valid for coprime positive integers u and p. Note that for such integers,

$$ur-ps>0 \iff ur>ps \iff \frac{ur}{p}>s \iff [\frac{ur}{p}]\geq s. \qquad \text{(A.2.1)}$$

In fact since $(u,p)=1$ and $0<r<p$, we need to deal with the integer part of $\frac{ur}{p}$ i.e. $[\frac{ur}{p}]$ rather than $\frac{ur}{p}$ itself, which is never an integer. So we concentrate on the

right-hand side of (2.4.17) and write,

$$\begin{aligned}
\sum_{r=1}^{p-1}\sum_{\substack{s=1\\ ur-ps\geq 1}}^{u-1} q^{ur-ps} &= \sum_{r=1}^{p-1} q^{ur} \sum_{s=1}^{[\frac{ur}{p}]} q^{-ps} , \\
&= \sum_{r=1}^{p-1} q^{ur-[\frac{ur}{p}]p} \sum_{s=1}^{[\frac{ur}{p}]} q^{([\frac{ur}{p}]-s)p} , \\
&= \sum_{r=1}^{p-1} q^{ur-[\frac{ur}{p}]p} \sum_{s=0}^{[\frac{ur}{p}]-1} q^{sp} ,
\end{aligned}$$

where in the last term we changed $s \to [\frac{ur}{p}] - s$. Now by using lemma (2.4.14) in the last summation above we obtain,

$$\begin{aligned}
\sum_{r=1}^{p-1} q^{ur-[\frac{ur}{p}]p} \cdot \frac{1-q^{[\frac{ur}{p}]p}}{1-q^p} &= \frac{\sum_{r=0}^{p-1} q^{ur-[\frac{ur}{p}]p} - \sum_{r=0}^{p-1} q^{ur}}{1-q^p} , \\
&= \frac{\sum_{t=0}^{p-1} q^{t} - \sum_{r=0}^{p-1} q^{ur}}{1-q^p} .
\end{aligned}$$

Since $ur - [\frac{ur}{p}]p \equiv ur \ mod(p)$ and as "r" runs from 0 to $(p-1)$, thus "t" runs from 0 to $(p-1)$ as well. So we again use (2.4.14) and obtain,

$$\sum_{t=0}^{p-1} q^t = \frac{1-q^p}{1-q} ,$$

and,

$$\sum_{r=0}^{p-1} q^{ur} = \frac{1-q^{up}}{1-q^u} .$$

Therefore taking these into account we finally arrive at,

$$\sum_{r=1}^{p-1}\sum_{\substack{s=1\\ ur-ps\geq 1}}^{u-1} q^{ur-ps} = \frac{1}{1-q} - \frac{1-q^{up}}{(1-q^u)(1-q^p)} ,$$

which is exactly the formula (2.4.17).

A.3 A rewriting of $\mathcal{K}_{(2p)}(q,x,y) - \mathcal{K}_{(2p)}(q,x^{-1},y)$

We start with the expression (2.4.20), which gives the level $2p$ Appell function as a sum of three terms t_1^+, t_2^+, t_3^+, each of which involves double (or triple) summations.

The superscript $^+$ (resp. $^-$) indicates that the second argument of $\mathcal{K}_{(2p)}$ is x, (resp. x^{-1}). We therefore write

$$\mathcal{K}_{(2p)}(q,x,y) - \mathcal{K}_{(2p)}(q,x^{-1},y) = t_1^+ - t_1^- + t_2^+ - t_2^- + t_3^+ - t_3^-, \tag{A.3.1}$$

and first concentrate on a rewriting of

$$\begin{aligned} t_1^+ - t_1^- &= \sum_{s=0}^{u-1}\sum_{b=0}^{u-1} x^{ps} y^{ps+2pb}\, q^{-pb^2-psb}\, \mathcal{K}_{(2p)}(q^{u^2}, x^u q^{\frac{us}{2}}, y^u q^{-\frac{us}{2}-bu}) \\ &\quad - \sum_{s=0}^{u-1}\sum_{b=0}^{u-1} x^{-ps} y^{ps+2pb}\, q^{-pb^2-psb}\, \mathcal{K}_{(2p)}(q^{u^2}, x^{-u} q^{\frac{us}{2}}, y^u q^{-\frac{us}{2}-bu}). \end{aligned} \tag{A.3.2}$$

Step 1: we show that $t_1^+ - t_1^-$ may be rewritten as the difference of double sums in (2.4.21) **modulo terms depending on the function ϑ introduced in** (2.2.3).

Note that the contributions $b=0$ and $b=u$ in t_1^+ only differ by a ϑ-term. This is easily seen when applying the periodicity property (2.4.10) to $\mathcal{K}_{(2p)}$ when $p \to 2p, q \to q^{u^2}, x \to x^u q^{\frac{us}{2}}, y \to y^u q^{-\frac{us}{2}}$ and $n=-1$. This allows us to write,

$$\begin{aligned} t_1^+ &= \sum_{s=0}^{u-1}\sum_{b=1}^{u} x^{ps} y^{ps+2pb}\, q^{-pb^2-pbs} \mathcal{K}_{(2p)}(q^{u^2}, x^u q^{\frac{us}{2}}, y^u q^{-\frac{us}{2}-bu}) \\ &\qquad + \sum_{s=0}^{u-1}\sum_{r=0}^{2p-1} x^{ps+ur} y^{ps+ur} \theta(q^{2pu^2}, x^{2pu} q^{u^2 r+usp}). \end{aligned} \tag{A.3.3}$$

On the other hand, relabel $s = u - s'$ in t_1^- and apply the periodicity property (2.4.11) to $\mathcal{K}_{(2p)}$ with $p \to 2p, q \to q^{u^2}, x \to x^{-u} q^{-\frac{us'}{2}}, y \to y^u q^{\frac{us'}{2}-bu}$ and $n=-p$ to arrive at,

$$\begin{aligned} t_1^- &= \sum_{s'=1}^{u}\sum_{b=0}^{u-1} x^{ps'} y^{-ps'+2pb}\, q^{-pb^2+pbs'} \mathcal{K}_{(2p)}(q^{u^2}, x^{-u} q^{-\frac{us'}{2}}, y^u q^{\frac{us}{2}-bu}) \\ &\quad - \sum_{s'=1}^{u}\sum_{b=0}^{u-1}\sum_{r=1-p}^{0} x^{ps'+ur} y^{2pb-ps'-ur} q^{-pb^2+pbs'+bur} \theta(q^{2pu^2}, x^{-2pu} q^{-u^2 r-pus'}) \\ &= t_{11}^- - t_{12}^-. \end{aligned} \tag{A.3.4}$$

We still need to manipulate t_{11}^- in order to complete step one of the proof.

• First note that the contributions $b=0$ and $b=u$ in t_{11}^{-} only differ by a ϑ-term. This is easily seen when applying the periodicity property (2.4.10) to $\mathcal{K}_{(2p)}$ when $p \to 2p, q \to q^{u^2}, x \to x^{-u}q^{-\frac{us'}{2}}, y \to y^u q^{\frac{us'}{2}}$ and $n=-1$. This allows us to write,

$$t_{11}^{-} = \sum_{s'=1}^{u}\sum_{b=1}^{u} x^{ps'}y^{-ps'+2pb}\, q^{-pb^2+pbs'}\mathcal{K}_{(2p)}(q^{u^2}, x^{-u}q^{-\frac{us'}{2}}, y^u q^{\frac{us'}{2}-bu})$$
$$+\sum_{s'=1}^{u}\sum_{r=0}^{2p-1} x^{ps'-ur}y^{-ps'+ur}\theta(q^{2pu^2}, x^{-2pu}q^{u^2r-ups}). \quad (A.3.5)$$

• Using the periodicity property (2.4.11) with $p \to 2p, q \to q^{u^2}, x \to x^{-u}, y \to y^u q^{-bu}$ and $n=p$, one sees that the contributions $s'=0$ and $s'=u$ differ by a θ-term in the first term of t_{11}^{-}. We have,

$$t_{11}^{-} = \sum_{s'=0}^{u-1}\sum_{b=1}^{u} x^{ps'}y^{-ps'+2pb}\, q^{-pb^2+pbs'}\mathcal{K}_{(2p)}(q^{u^2}, x^{-u}q^{-\frac{us'}{2}}, y^u q^{\frac{us'}{2}-bu})$$
$$+\sum_{b=1}^{u}\sum_{r=1}^{p} x^{ur}y^{-ur+2pb}q^{-pb^2+bur}\theta(q^{2pu^2}, x^{-2pu}q^{-u^2r})$$
$$+\sum_{s'=1}^{u}\sum_{r=0}^{2p-1} x^{ps'-ur}y^{-ps'+ur}\theta(q^{2pu^2}, x^{-2pu}q^{u^2r-ups}). \quad (A.3.6)$$

• Finally, we rewrite the first term in t_{11}^{-} as

$$\sum_{s'=0}^{u-1}\sum_{b=1}^{u} f(s',b), \quad (A.3.7)$$

where

$$f(s',b) = x^{ps'}y^{-ps'+2pb}\, q^{-pb^2+pbs'}\mathcal{K}_{(2p)}(q^{u^2}, x^{-u}q^{-\frac{us'}{2}}, y^u q^{\frac{us'}{2}-bu}) \quad (A.3.8)$$

and note that

$$\sum_{s'=0}^{u-1}\sum_{b=1}^{u} f(s',b) = \sum_{s'=0}^{u-1}\sum_{b=1}^{u} f(s',b+s') + \sum_{s'=1}^{u-1}\sum_{b=1}^{s'}[f(s',b) - f(s',b+u)]. \quad (A.3.9)$$

Applying the periodicity property (2.4.10) to $\mathcal{K}_{(2p)}$ when $p \to 2p, q \to q^{u^2}, x \to$

$x^{-u}q^{-\frac{us'}{2}}, y \to y^u q^{\frac{us'}{2}-bu}$ and $n=-1$, we arrive at,

$$\begin{aligned}
t_{11}^{-} = &\sum_{s'=0}^{u-1}\sum_{b=1}^{u} x^{ps'}y^{ps'+2pb}\,q^{-pb^2-pbs'}\mathcal{K}_{(2p)}(q^{u^2}, x^{-u}q^{-\frac{us'}{2}}, y^u q^{-\frac{us'}{2}-bu})\\
&+\sum_{s'=1}^{u-1}\sum_{b=1}^{s'}\sum_{r=0}^{2p-1} x^{ps'-ur}y^{-ps'+ur+2pb}q^{-pb^2+ps'b-bur}\theta(q^{2pu^2}, x^{-2pu}q^{u^2r-us'p})\\
&\qquad+\sum_{b=1}^{u}\sum_{r=1}^{p} x^{ur}y^{-ur+2pb}q^{-pb^2+bur}\theta(q^{2pu^2}, x^{-2pu}q^{-u^2r})\\
&\qquad\qquad+\sum_{s'=1}^{u}\sum_{r=0}^{2p-1} x^{ps'-ur}y^{-ps'+ur}\theta(q^{2pu^2}, x^{-2pu}q^{u^2r-ups}). \qquad (A.3.10)
\end{aligned}$$

In conclusion, we have, after collecting the information in (A.3.3), (A.3.4) and (A.3.10),

$$\begin{aligned}
t_1^{+} - t_1^{-} = &\sum_{s=0}^{u-1}\sum_{b=1}^{u} x^{ps}y^{ps+2pb}\,q^{-pb^2-pbs}\\
&\times[\mathcal{K}_{(2p)}(q^{u^2}, x^u q^{\frac{su}{2}}, y^u q^{-\frac{su}{2}-bu}) - \mathcal{K}_{(2p)}(q^{u^2}, x^{-u}q^{-\frac{su}{2}}, y^u q^{-\frac{su}{2}-bu})]+\\
&\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5, \qquad (A.3.11)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_1 &= \sum_{s=0}^{u-1}\sum_{r=0}^{2p-1} x^{ps+ur}y^{ps+ur}\theta(q^{2pu^2}, x^{2pu}q^{u^2r+usp}),\\
\mathcal{T}_2 &= \sum_{s'=1}^{u}\sum_{b=0}^{u-1}\sum_{r=1-p}^{0} x^{ps'+ur}y^{2pb-ps'-ur}q^{-pb^2+pbs'+bur}\theta(q^{2pu^2}, x^{-2pu}q^{-u^2r-pus'}),\\
\mathcal{T}_3 &= -\sum_{b=1}^{u}\sum_{r=1}^{p} x^{ur}y^{-ur+2pb}q^{-pb^2+bur}\theta(q^{2pu^2}, x^{-2pu}q^{-u^2r}),\\
\mathcal{T}_4 &= -\sum_{s'=1}^{u}\sum_{r=0}^{2p-1} x^{ps'-ur}y^{-ps'+ur}\theta(q^{2pu^2}, x^{-2pu}q^{u^2r-ups}),\\
\mathcal{T}_5 &= -\sum_{s'=1}^{u-1}\sum_{b=1}^{s'}\sum_{r=0}^{2p-1} x^{ps'-ur}y^{-ps'+ur+2pb}q^{-pb^2+ps'b-bur}\theta(q^{2pu^2}, x^{-2pu}q^{u^2r-us'p})
\end{aligned}$$
$$(A.3.12)$$

are the θ-terms emerging from (A.3.2) after repeated use of the periodicity properties of the level $2p$ Appell function.

Step 2: we indicate how all terms in (A.3.12) **with a negative power of** y **add up to zero.**

The term

$$\sum_{\substack{b=1-u \\ 2pb+ps-ur>0}}^{u} \sum_{r=1}^{2p-1} \sum_{s=0}^{u-1} x^{ps-ur} y^{2pb+ps-ur} q^{-pb^2-pbs+bur} \Lambda_{(r,s+1,u,p)}(q^u, x^{-2u}) \tag{A.3.13}$$

in (2.4.21) only contains positive powers of y, as do the terms $t_2^{\pm}$ and $t_3^{\pm}$ in (A.3.1). We therefore analyse the negative y-powers in the θ-terms appearing in (A.3.12). They potentially appear in $\mathcal{T}_i, i = 2,3,4,5$.

- Change variable from $r \to -r$ in $\mathcal{T}_2$ and add it to $\mathcal{T}_4$ to obtain

$$\begin{aligned}\mathcal{T}_2 + \mathcal{T}_4 = &\sum_{s=1}^{u} \sum_{b=1}^{u-1} \sum_{r=0}^{p-1} x^{ps-ur} y^{2pb-ps+ur} q^{-pb^2+pbs-bur} \theta(q^{2pu^2}, x^{-2pu} q^{u^2 r - ups}) \\ &- \sum_{s=1}^{u} \sum_{r=p}^{2p-1} x^{ps-ur} y^{-ps+ur} \theta(q^{2pu^2}, x^{-2pu} q^{u^2 r - ups}) \\ &\equiv \mathcal{T}_2' + \mathcal{T}_4'.\end{aligned} \tag{A.3.14}$$

- Now change variable from $r \to -r$ in $\mathcal{T}_3$ and add it to $\mathcal{T}_2'$. This yields

$$\begin{aligned}\mathcal{T}_2' + \mathcal{T}_3 = &\sum_{s=1}^{u} \sum_{b=1}^{u-1} \sum_{r=0}^{p-1} x^{ps-ur} y^{2pb-ps+ur} q^{-pb^2+pbs-bur} \theta(q^{2pu^2}, x^{-2pu} q^{u^2 r - ups}) \\ &- \sum_{b=1}^{u} \sum_{r=-p}^{-1} x^{-ur} y^{ur+2pb} q^{-pb^2-bur} \theta(q^{2pu^2}, x^{-2pu} q^{u^2 r}).\end{aligned} \tag{A.3.15}$$

After a succession of standard manipulations on the above sums, one arrives at the following expression,

$$\begin{aligned}\mathcal{T}_2' + \mathcal{T}_3 = &\sum_{s=1}^{u-1} \sum_{b=1}^{u-1} \sum_{r=0}^{p-1} x^{ps-ur} y^{2pb-ps+ur} q^{-pb^2+pbs-bur} \theta(q^{2pu^2}, x^{-2pu} q^{u^2 r - ups}) \\ &- \sum_{r=1}^{p} x^{ur} y^{u(2p-r)} q^{(r-p)u^2} \theta(q^{2pu^2}, x^{-2pu} q^{-u^2 r}).\end{aligned} \tag{A.3.16}$$

- Consider the contribution

$$\begin{aligned}\mathcal{T}_4' + \mathcal{T}_5 = &- \sum_{s=1}^{u} \sum_{r=p}^{2p-1} x^{ps-ur} y^{-ps+ur} \theta(q^{2pu^2}, x^{-2pu} q^{u^2 r - ups}) \\ &- \sum_{s=1}^{u-1} \sum_{b=1}^{s} \sum_{r=0}^{2p-1} x^{ps-ur} y^{-ps+ur+2pb} q^{-pb^2+psb-bur} \theta(q^{2pu^2}, x^{-2pu} q^{u^2 r - usp})\end{aligned} \tag{A.3.17}$$

which, after a few standard manipulations of the sums, yields

$$\begin{aligned}
\mathcal{T}_4' + \mathcal{T}_5 = &-\sum_{s=1}^{u-1}\sum_{b=1}^{u-1}\sum_{r=0}^{p-1} x^{ps-ur}y^{2pb-ps+ur}q^{-pb^2+pbs-bur}\theta(q^{2pu^2},x^{-2pu}q^{u^2r-ups}) \\
&-\sum_{s=1}^{u-1}\sum_{b=0}^{u-1}\sum_{r=p}^{2p-1} x^{ps-ur}y^{2pb-ps+ur}q^{-pb^2+pbs-bur}\theta(q^{2pu^2},x^{-2pu}q^{u^2r-ups}) \\
&+\sum_{s=1}^{u-1}\sum_{b=s+1}^{u-1}\sum_{r=0}^{2p-1} x^{ps-ur}y^{2pb-ps+ur}q^{-pb^2+pbs-bur}\theta(q^{2pu^2},x^{-2pu}q^{u^2r-ups}) \\
&-\sum_{r=0}^{p-1} x^{-ur}y^{ur}\theta(q^{2pu^2},x^{-2pu}q^{u^2r}). \qquad (A.3.18)
\end{aligned}$$

• Finally, putting everything together, we get,

$$\begin{aligned}
&\mathcal{T} \equiv \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 = \mathcal{T}_2' + \mathcal{T}_3 + \mathcal{T}_4' + \mathcal{T}_5 = \\
&-\sum_{s=1}^{u-1}\sum_{b=0}^{u-1}\sum_{r=0}^{p-1} x^{-ur-p(u-s)}y^{2pb+ur+p(u-s)}q^{-pb^2-pb(u-s)-bur}\theta(q^{2pu^2},x^{-2pu}q^{u^2(r+p)-ups}) \\
&-\sum_{r=1}^{p} x^{ur}y^{u(2p-r)}q^{(r-p)u^2}\theta(q^{2pu^2},x^{-2pu}q^{-u^2r}) \\
&+\sum_{s=1}^{u-1}\sum_{b=s+1}^{u-1}\sum_{r=0}^{2p-1} x^{ps-ur}y^{2pb-ps+ur}q^{-pb^2+pbs-bur}\theta(q^{2pu^2},x^{-2pu}q^{u^2r-ups}) \\
&-\sum_{r=0}^{p-1} x^{-ur}y^{ur}\theta(q^{2pu^2},x^{-2pu}q^{u^2r}), \qquad (A.3.19)
\end{aligned}$$

and indeed, all negative powers in y have disappeared.

Step 3: reorganising finite sums

Our final goal is to re-express the sum

$$\mathcal{T}_1 + \mathcal{T} + t_2^+ - t_2^- + t_3^+ - t_3^- \qquad (A.3.20)$$

as (A.3.13), with

$$\Lambda_{(r,s+1,u,p)}(q,x) = \theta(q^{2pu},x^pq^{ur-p(s-1)}) - q^{r(s-1)}x^{-r}\theta(q^{2pu},x^pq^{-ur-p(s-1)}). \qquad (A.3.21)$$

This step is most easily completed with Maple, using a power expansion in the variable q, checking one obtains an identity for each power of q, with a wide number of sample values for the two coprimes u and p.

A.4 A relation between Appell and theta functions

We prove that the remarkable identity (2.4.26) can easily be derived from an identity by Kac [11]. We start with

$$\sum_{m\in\mathbb{Z}}\mathcal{K}_{(p)}(q,z,yq^m)x^m=\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{q^{n^2p/2}z^{np}x^m}{1-zyq^{m+n}}=\sum_{n\in\mathbb{Z}}q^{n^2p/2}z^{np}\sum_{m\in\mathbb{Z}}\frac{x^{m-n}}{1-zyq^m}, \qquad (A.4.1)$$

where in the last equality we have changed $m\to m-n$. Therefore we continue as,

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\mathcal{K}_{(p)}(q,z,yq^m)x^m &= \sum_{n\in\mathbb{Z}}q^{n^2p/2}z^{np}x^{-n}\sum_{m\in\mathbb{Z}}\frac{x^m}{1-zyq^m}\\ &= \theta(q^p,z^px^{-1})\sum_{m\in\mathbb{Z}}\frac{x^m}{1-zyq^m}. \qquad (A.4.2)\end{aligned}$$

At this point we separately need to show that,

$$\sum_{m\in\mathbb{Z}}\frac{z^m}{1-yq^m}=-\frac{\vartheta_{(1,1)}(q,zy)\prod_{i\geq 1}(1-q^i)^3}{\vartheta_{(1,1)}(q,y)\vartheta_{(1,1)}(q,z)}\,. \qquad (A.4.3)$$

To do that we call an identity from [11], namely,

$$\sum_{j\in\mathbb{Z}}\frac{(-x)^j}{1+yq^j}=\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n)}{(1+xq^{n-1})(1+x^{-1}q^n)(1+yq^{n-1})(1+y^{-1}q^n)}\,.$$

Now by changing $x\to -z$ and $y\to -y$, one obtains,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\frac{z^j}{1-yq^j} &= \frac{\prod_{n=1}^{\infty}(1-q^n)^3\prod_{n=1}^{\infty}(1-q^n)(1-zyq^{n-1})(1-z^{-1}y^{-1}q^n)}{\prod_{n=1}^{\infty}(1-q^n)(1-zq^{n-1})(1-z^{-1}q^n)\cdot(1-q^n)(1-yq^{n-1})(1-y^{-1}q^n)}\\ &= \frac{\vartheta_{(1,1)}(q,z^{-1}y^{-1})\prod_{n=1}^{\infty}(1-q^n)^3}{\vartheta_{(1,1)}(q,z^{-1})\vartheta_{(1,1)}(q,y^{-1})},\\ &= -\frac{\vartheta_{(1,1)}(q,zy)\prod_{n=1}^{\infty}(1-q^n)^3}{\vartheta_{(1,1)}(q,z)\vartheta_{(1,1)}(q,y)}, \qquad (A.4.4)\end{aligned}$$

where $\vartheta_{(1,1)}(q,z^{-1})=-z\,\vartheta_{(1,1)}(q,z)$. In (A.4.4) indeed, we obtained what we claimed in (A.4.3). Now, by changing $z\to x$ and $y\to zy$ in (A.4.3) and putting the result in (A.4.1) we deduce,

$$\sum_{m\in\mathbb{Z}}\mathcal{K}_{(p)}(q,z,yq^m)x^m=-\theta(q^p,z^px^{-1})\frac{\vartheta_{(1,1)}(q,zyx)\,q^{\frac{-1}{8}}\tilde{\eta}(q)^3}{\vartheta_{(1,1)}(q,zy)\vartheta_{(1,1)}(q,x)}\,, \qquad (A.4.5)$$

which is exactly the formula (2.4.26). Note however the special limitations of q and x mentioned in [11], causing the formula (A.4.5) to be only valid for $|q|<|x|<1$.

Appendix B

B.1 A remarkable identity

To extract the relation (3.4.12), valid for u and p positive coprime integers, we have similar arguments as we had in (A.2.1) and instantly try to express the right-hand side of (3.4.12) as,

$$
\begin{aligned}
-\sum_{\substack{r=1\\ ur-ps>0}}^{p-1}\sum_{s=1}^{u-1} q^{ps-ur} &= -\sum_{r=1}^{p-1} q^{-ur}\sum_{s=1}^{[\frac{ur}{p}]} q^{ps} , \\
&= -\sum_{r=1}^{p-1} q^{-ur+[\frac{ur}{p}]p}\sum_{s=0}^{[\frac{ur}{p}]-1} q^{-sp} , \\
&= -\sum_{r=1}^{p-1} q^{-ur+[\frac{ur}{p}]p} . \frac{1-q^{-[\frac{ur}{p}]p}}{1-q^{-p}} ,
\end{aligned}
\tag{B.1.1}
$$

recalling that,

$$
\sum_{r=1}^{p-1} q^{-(ur-[\frac{ur}{p}]p)} = \sum_{t=1}^{p-1} q^{-t} = \frac{1-q^{-p}}{1-q^{-1}} \text{ and, } \sum_{\mathrm{r}=0}^{\mathrm{p}-1} \mathrm{q}^{-\mathrm{ur}} = \frac{1-\mathrm{q}^{-\mathrm{up}}}{1-\mathrm{q}^{-\mathrm{u}}} .
$$

Put these two fractions in (B.1.1) we get,

$$
-\sum_{\substack{r=1\\ ur-ps>0}}^{p-1}\sum_{s=1}^{u-1} q^{ps-ur} = \frac{(1-q^{-up})(1-q^{-1})-(1-q^{-p})(1-q^{-u})}{(1-q^{-1})(1-q^{-u})(1-q^{-p})} . \tag{B.1.2}
$$

Furthermore, a very small calculation for the left-hand side of (3.4.12) shows that,

$$
\sum_{r=1}^{2p}\sum_{s=1}^{u} \frac{q^{pu-u(r-1)-p(s-1)}}{q^{pu}-q^{-pu}} = \frac{q^{up}}{q^{up}-q^{-up}}\Big(\sum_{r=0}^{2p-1} q^{-ur}\Big)\Big(\sum_{s=0}^{u-1} q^{-ps}\Big) , \tag{B.1.3}
$$

in which by using the lemma (2.4.14) again, we reach the same result as in (B.1.1). Hence we have properly proved the formula (3.4.12).

Appendix C

C.1 Period-increasing of $F(\mu,\nu,\tau)$

To prove formula (4.3.10) as a period increasing statement for the function $F(\mu,\nu,\tau)$, we start by writing the F function given in (4.3.6) in terms of the remarkable identity introduced in (2.4.26), namely

$$F(\mu,\nu,\tau) = -\sum_{m\in\mathbb{Z}} \frac{e^{2i\pi m(\nu+\frac{1}{2})}}{1-e^{-2i\pi(\mu+\frac{1}{2})}e^{2i\pi m\tau}}. \tag{C.1.1}$$

The F function appearing in the right-hand side of (4.3.10) becomes,

$$F(u\alpha+\tau a+\tfrac{1+u}{2}, u\beta-\tau b+\tfrac{1+u}{2}, u\tau) = -\sum_{m\in\mathbb{Z}} \frac{e^{2i\pi m(\beta u - b\tau + \frac{u}{2})}}{1-e^{-2i\pi(\alpha u + a\tau + \frac{u}{2})}e^{2i\pi m u\tau}} \tag{C.1.2}$$

and the right-hand side of (4.3.10) will therefore be given by,

$$[R.H.S] = -\sum_{m\in\mathbb{Z}}\sum_{a,b=0}^{u-1} \frac{e^{2i\pi\beta(mu-a)-2i\pi b\frac{\tau}{u}(mu-a)+i\pi(mu-a)}e^{2i\pi b(\alpha+\frac{1}{2})}}{1-e^{2i\pi\tau(mu-a)}e^{-2i\pi u(\alpha+\frac{1}{2})}}. \tag{C.1.3}$$

Now using (2.4.12), the double summation above reduces to,

$$\begin{aligned}
[R.H.S] &= -\sum_{m\in\mathbb{Z}} \frac{e^{2i\pi\beta m + m i\pi}}{1-e^{2i\pi m\tau}e^{-2i\pi u(\alpha+\frac{1}{2})}} \sum_{b=0}^{u-1} e^{b[-2i\pi m\frac{\tau}{u}+2i\pi(\alpha+\frac{1}{2})]} \\
&= -e^{2i\pi(\alpha+\frac{1}{2})(u-1)} \sum_{m\in\mathbb{Z}} \frac{e^{2i\pi m[\beta+\frac{1}{2}-\tau(1-\frac{1}{u})]}}{1-e^{2i\pi m\frac{\tau}{u}}e^{-2i\pi(\alpha+\frac{1}{2})}} \\
&= e^{2i\pi(\alpha+\frac{1}{2})(u-1)} \frac{\prod_{j\in\mathbb{N}}(1-e^{2i\pi\frac{\tau}{u}j})^3\vartheta_{1,1}(\frac{\tau}{u},\beta-\alpha-(u-1)\frac{\tau}{u})}{\vartheta_{1,1}(\frac{\tau}{u},-\alpha-\frac{1}{2})\vartheta_{1,1}(\frac{\tau}{u},\beta+\frac{1}{2}-(u-1)\frac{\tau}{u})},
\end{aligned} \tag{C.1.4}$$

where in the second line, the elementary lemma (2.4.14), and in the third line the remarkable identity (2.4.26) have been used. Using (2.2.13) and selecting $\theta = 1-u$, one can rewrite

$$\vartheta_{1,1}(\frac{\tau}{u}, \beta - \alpha - (u-1)\frac{\tau}{u}) = (-1)^{1-u} e^{-i\pi\frac{\tau}{u}[(1-u)^2+(1-u)]} e^{2i\pi(\beta-\alpha)(u-1)} \vartheta_{1,1}(\frac{\tau}{u}, \beta - \alpha),$$

and,

$$\vartheta_{1,1}(\frac{\tau}{u}, \beta + \frac{1}{2} - (u-1)\frac{\tau}{u}) = (-1)^{1-u} e^{-i\pi\frac{\tau}{u}[(1-u)^2+(1-u)]} e^{2i\pi(\beta+\frac{1}{2})(u-1)} \vartheta_{1,1}(\frac{\tau}{u}, \beta + \frac{1}{2}).$$

Implementing these two rewritten items in (C.1.4), we finally arrive at,

$$[R.H.S] = \frac{\prod_{j\in\mathbb{N}}(1 - e^{2i\pi\frac{\tau}{u}j})^3 \vartheta_{1,1}(\frac{\tau}{u}, \beta - \alpha)}{\vartheta_{1,1}(\frac{\tau}{u}, -\alpha - \frac{1}{2})\vartheta_{1,1}(\frac{\tau}{u}, \beta + \frac{1}{2})} = F(\alpha, \beta, \frac{\tau}{u}) \,, \tag{C.1.5}$$

which is exactly the left-hand side of (4.3.10).

We also quote how the function F transforms under the modular group. Its behaviour under the generating transformations S and T is easily derived using (2.2.18), (2.2.19) and (2.2.24) as well as by remarking that $\vartheta_{1,1}$ and $\vartheta_{1,0}$ are invariant under T. We have,

$$T.F = F(\mu, \nu, \tau + 1) = F(\mu, \nu, \tau) \tag{C.1.6}$$

and

$$S.F = F(\frac{\mu}{\tau}, \frac{\nu}{\tau}, \frac{-1}{\tau}) = \tau e^{i\pi(\mu+\nu) - \frac{i\pi}{2}\tau - 2i\pi\frac{\mu\nu}{\tau}} F(\mu - \frac{\tau}{2} + \frac{1}{2}, \nu - \frac{\tau}{2} + \frac{1}{2}, \tau) \,. \tag{C.1.7}$$

C.2 Definition and some identities for $\Lambda_{(r,s,u,p)}(q,x)$

$\Lambda_{(r,s,u,p)}$ has been defined in (C.2.1). It is expressed via theta functions as,

$$\Lambda_{(r,s,u,p)}(q,x) = \theta(q^{2pu}, x^p q^{ur-p(s-1)}) - q^{r(s-1)} x^{-r} \theta(q^{2pu}, x^p q^{-ur-p(s-1)}). \quad \text{(C.2.1)}$$

We occasionally use instead the notation,

$$\Lambda_{r,s,u,p}(\tau,\nu) = \vartheta(2pu\tau, p\nu - p(s-1)\tau + ur\tau) \\ - e^{2i\pi r(s-1)\tau - 2i\pi r\nu} \vartheta(2pu\tau, p\nu - p(s-1)\tau - ur\tau). \quad \text{(C.2.2)}$$

For this function it simply follows that,

$$\Lambda_{0,s,u,p} = \Lambda_{p,s,u,p} = \Lambda_{2p,s,u,p} = 0\ . \quad \text{(C.2.3)}$$

Among other "shift-reflection" identities satisfied by $\Lambda_{r,s,u,p}$, we note ($n \in \mathbb{Z}$)

$$\Lambda_{2pn-r,s,u,p}(\tau,\nu) = -e^{-2i\pi(np-r)(\nu-(s-1)\tau+nu\tau)} \Lambda_{r,s,u,p}(\tau,\nu), \quad \text{(C.2.4)}$$

$$\Lambda_{r+p,s-u,u,p}(\tau,\nu) = e^{-2i\pi p\nu - 2i\pi ur'\tau + 2i\pi p(s'-1)\tau - 2i\pi pu\tau} \Lambda_{r,s,u,p}(\tau,\nu). \quad \text{(C.2.5)}$$

We should note that, all the labels used above, have been arranged to serve specially the $\widehat{s\ell}(2|1)$ non-unitary case, however just a change of $r \leftrightarrow s$ or $r' \leftrightarrow s'$ must make everything ready to be used for $N = 2$ characters.

C.3 a collected formula from what was done while calculating the $\mathcal{H}^-$

If we precisely follow the calculation we had for $\mathcal{H}^-$ in chapter 4 and try to simplify the common factors in front of involved terms, $\mathcal{H}_1^-$ in (4.2.44) and $\mathcal{H}_2^-$ from (4.2.50), we could describe the following formula as an individual identity as,

$$\sum_{a=0}^{2p-1} e^{i\pi\frac{ua^2}{2p\tau}+2i\pi a\frac{\gamma}{\tau}} \Phi(\tfrac{\tau}{2pu}, \tfrac{\gamma}{u}+\tfrac{a}{2p})\, \vartheta(\tfrac{\tau}{2pu}, \tfrac{\eta}{u}-\tfrac{a}{2p}) =$$

$$\begin{aligned}
&= 2p \sum_{s'=1}^{2p} \sum_{r'=1}^{u} \sum_{\substack{n\in\mathbb{Z} \\ 1\le r'+2n\le 2u}} e^{2i\pi\frac{\eta}{u}(u(s'-1)+p(r'+2n-1))+i\pi\frac{\tau}{2pu}(u(s'-1)+p(r'+2n-1))^2} \\
&\qquad \times \Phi(2pu\tau, 2p\gamma - u(s'-1)\tau - p(r'-1)\tau) \\
&\qquad \times \vartheta(2pu\tau, 2p\eta + u(s'-1)\tau + p(r'+2n-1)\tau) \\
&\quad - 2p\, e^{-2i\pi p\frac{\gamma^2}{u\tau}} \sum_{r'=1}^{u-1} \sum_{\substack{n\in\mathbb{Z} \\ 1-2u\le r'+2n\le 0 \\ s'u-r'p>0}} \sum_{s'=1}^{p-1} e^{2i\pi(s'-\frac{p}{u}r')\gamma - 2i\pi n[s'-\frac{p}{u}(r'+n)]\tau} \\
&\qquad \times e^{2i\pi\eta(s'-\frac{p}{u}(r'+2n))} \vartheta(2pu\tau, 2p\eta + s'u\tau - p(r'+2n)\tau). \qquad \text{(C.3.6)}
\end{aligned}$$

By doing that, in fact we have a good possibility to use it as a very ready identity for chapter 5 as well.

Appendix D

D.1 A period-increasing formula for ϑ functions and related discussions

To prove the period-increasing formula (4.2.33), we first prove the following simple theorem.

Theorem If the function $f(n)$ for $n \in \mathbb{Z}$ is periodic of period $2pu$, with $(u,p) = 1$, i.e. if

$$f(n) = f(n + 2pul), \qquad l \in \mathbb{Z} \ , \tag{D.1.1}$$

one can show,

$$\sum_{n=0}^{2pu-1} f(n) = \sum_{r=0}^{2p-1} \sum_{s=0}^{u-1} f(\pm ru \pm sp) \ . \tag{D.1.2}$$

Proof Indeed since $2pu$ is the period of f, we select two pairs of integers r, s and r', s' and write,

$$\pm ru \pm sp \equiv \pm r'u \pm s'p \pmod{2pu} \ .$$

Therefore,

$$\pm(r - r')u \equiv \pm(s' - s)p \pmod{2pu} \ ,$$

and since $(u,p) = 1$, $u|(s' - s)$ and $0 \leq s, s' < u \ \Rightarrow \ s = s'$,
which implies,

$$\pm(r - r')u \equiv \pm(r - r') \equiv 0 \pmod{2pu} \ .$$

Also $0 \leq r, r' < 2p \;\Rightarrow\; r = r'$, which means that any pair of integers r, s in the ranges given in (D.1.2) contributes to the double summation in the right hand side of (D.1.2).

Now take $f(n) = e^{i\pi \frac{\tau}{2pu} n^2} e^{2i\pi\mu n} \vartheta(2pu\tau, n + 2pu\mu)$—which readily satisfies (D.1.1)—and use the identity (D.1.2) to obtain,

$$\vartheta(\tfrac{\tau}{2pu}, \tfrac{\nu}{u} - \tfrac{a}{2p}) = \sum_{r''=1}^{2p} \sum_{s''=1}^{u} e^{2i\pi(\frac{\nu}{u} - \frac{a}{2p})[u(r''-1)+p(s''-1)] + i\pi\frac{u\tau}{2p}[r''-1+\frac{p}{u}(s''-1)]^2}$$

$$\times\, \vartheta(2pu\tau, 2p\nu + u(r''-1)\tau + p(s''-1)\tau), \quad \text{(D.1.3)}$$

in which we have used (2.4.13), taken $\mu = \frac{\nu}{u} - \frac{a}{2p}$, and have selected plus signs for both r'' and s'' indices. A simple change of $r'' \leftrightarrow s''$ yields the formula in (4.2.33).

In addition there is another useful property of sums of the type (D.1.2) and (D.1.3) to note. In (D.1.2), the double summation can be replaced by

$$\sum_{r=0}^{p-1} \sum_{s=0}^{2u-1} . \quad \text{(D.1.4)}$$

This is a direct consequence of f having period $2up$. The summand in (D.1.3) is a particular case of function f, with argument depending on $u(r''-1) + p(s''-1)$. There, we can change

$$\sum_{s''=1}^{2p} \sum_{r''=1}^{u} \to \sum_{s''=1}^{p} \sum_{r''=1}^{2u} \quad \text{(D.1.5)}$$

D.2 Some further identities for Chapters 4 and 5

An obvious, but still useful to note, is the following identity,

$$\sum_{s'=1}^{2u} \sum_{\substack{n \\ 1 \leq s'+2n \leq 2u}} f(s'+2n, s') = \sum_{s'=1}^{2u} \sum_{\substack{n \\ 1 \leq s'+2n \leq 2u}} f(s', s'+2n) \,. \quad \text{(D.2.6)}$$

Moreover, in the right hand side of formula above, summing separately over even and odd values of s' gives,

$$\sum_{\substack{s'=1 \\ \text{even} \\ [n\to n-\frac{s'}{2}]}}^{2u} \sum_{\substack{n \\ 1\le 2n\le 2u}} f(s',2n) + \sum_{\substack{s'=1 \\ \text{odd} \\ [n\to n-\frac{s'+1}{2}]}}^{2u} \sum_{\substack{n \\ 1\le 2n-1\le 2u}} f(s',2n-1) =$$

$$= \sum_{n=1}^{u}\Big(\sum_{\substack{s'=1 \\ \text{even}}}^{2u} f(s',2n) + \sum_{\substack{s'=1 \\ \text{odd}}}^{2u} f(s',2n-1)\Big) = \sum_{n=1}^{u}\sum_{s'=1}^{2u} f(s',2n-[s']_2) \ , \quad \text{(D.2.7)}$$

with $[s]_2 = s \bmod 2$.

Furthermore an easy to derive, but indeed a truly fruitful identity is,

$$\sum_{s'=1}^{u}\sum_{\substack{n\in\mathbb{Z} \\ 1\le s'+2n\le 2u}} \Psi(s')f(s'+2n) = \sum_{s'=1}^{u}\Psi(s')\sum_{b=1}^{u} f(2b-[s']_2) \ , \qquad \text{(D.2.8)}$$

where $\Psi(s')$ is an arbitrary function of s'. The proof is also totally similar to (D.2.7). Formula above could be reformed as,

$$\sum_{s'=1}^{u}\sum_{\substack{n\in\mathbb{Z} \\ 1-2u\le s'+2n\le 0}} \Psi(s')f(s'+2n) = \sum_{s'=1}^{u}\Psi(s')\sum_{b=1-u}^{0} f(2b-[s']_2) \ , \qquad \text{(D.2.9)}$$

however worth noting that, any similar identity same as above equalities still hold if, $a \le s'+2n \le b$ for any integer a and b, as long as $b-a = 2u-1$. It can also be shown that for special kinds of function f appearing in the identity (D.2.8), namely function f such that,

$$f(2u+1) = f(1) \ , \qquad \text{(D.2.10)}$$

we can extend the validity of (D.2.8)(for (D.2.9) as well) to,

$$\sum_{s'=1}^{u}\sum_{\substack{n\in\mathbb{Z} \\ 1\le s'+2n\le 2u}} \Psi(s')f(s'+2n) = \sum_{s'=1}^{u}\Psi(s')\sum_{b=1}^{u} f(2b\pm[s']_2) \ . \qquad \text{(D.2.11)}$$

Obviously the same extend of validity can be also achieved for (D.2.9).

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